COMP 280: Assignment 4, Sample Solutions

1. (cons boolsym boolsym)
   An Answer is either:
   - a boolean
   - a boolsym.

(define (check-equi-vars F1 F2 vars)
  (local ((define (check-equi-vars-acc F1 F2 vars acc)
   (cond [(empty? vars) (if (equal? (eval F1) (eval F2))
     #t acc))
   [else (let ((answer-true (check-equi-vars-acc (subst 'true (first vars) F1)
     (subst 'true (first vars) F2)
     (rest vars)
     (append acc (list 'true))))
     (answer-false (check-equi-vars-acc (subst 'false (first vars) F1)
     (subst 'false (first vars) F2)
     (rest vars)
     (append acc (list 'false)))))))
   (cond
     [(and (boolean? answer-true) (boolean? answer-false)) #t]
     [(list? answer-false) answer-false]
     [(list? answer-true) answer-true]])))))

2. (a) ∃ student : in-class(student) ∧ ¬ like(student, owls).
    There is a student in this class who dislikes owls.
   (b) ∀ student : in-class(student) → has-been(student, Sammy’s).
    All students in this class have been to Sammy’s.
   (c) ∀ student : in-class(student) → ∃ course : math(course) ∧ Rice(course) ∧ ¬ taken(student, course).
    Every student in this class has not taken at least one Rice mathematics course.
   (d) ∃ student : in-class(student) ∧ ∃ building : on-campus(building) ∧ ∀ room : in-building(room, building) → ¬ has-been(student, room).
    There is a student in this class who has not visited any room of some building on campus.

3. (a) Model: let student and prof be natural numbers n and m, and class-from(n, m) be (n * m) = 0.
    Then the statement becomes:
\(\forall n \exists m \colon (n \cdot m) = 0 \rightarrow \exists n \forall m \colon (n \cdot m) = 0\)

which is true, since both sides are true (pick zero for \(m\)).

Countermodel: let student and prof be natural numbers \(n\) and \(m\), and class-from be equality. Then the statement becomes:

\(\forall n \exists m \colon n = m \rightarrow \exists n \forall m \colon n = m\)

which is false, since the left part of the statement is true (take \(m\) to be \(n\)) and the right part is false (or else all the natural numbers would have to be equal).

(b) Model: let bar be a natural number \(n\), ball be zero, baz(\(m\)) be \((m + 0)\), and foo(\(n, m\)) be \(n \geq m\).

Then the statement becomes:

\(\forall n \colon n \geq (0 + 0) \rightarrow \forall n \colon n \geq 0\)

Countermodel: let bar be a natural number \(n\), ball be 17, baz(\(m\)) be \((m - 17)\), and foo(\(n, m\)) be \(n \geq m\). Then the statement becomes:

\(\forall n \colon n \geq (17 - 17) \rightarrow \forall n \colon n \geq 17\)

which is obviously false.

Extra credit: see the model given above.

4. (a) "\(\forall x \forall y\)" means "for all \(x\) and for all \(y\)", which is the same as "for all \(y\) and for all \(x\)".

(b) "\(\exists y \forall x\)" means "for all \(x\) there is at least one \(y\) such that \(\ldots \)", but the \(y\) can depend on the value of \(x\), and be different each time.

On the other hand, "\(\forall y \exists x\)" means "there is at least one \(y\) such that for all \(x \ldots \)". So the same \(y\) works for all possible values of \(x\).

So the two parts of the statement do not have the same meaning.

5. If \(I\) be a model for \(\exists x : (P(x) \lor Q(x))\) with domain \(D\), then \(\exists x : (P(x) \lor Q(x))\) is true for \(I\), which means that \((P(d) \lor Q(d))\) is true for some \(d \in D\). Therefore, at least one of \(P(d)\) and \(Q(d)\) is true for some \(d \in D\). So at least one of \(\exists x : P(x)\) and \(\exists x : Q(x)\) is true for \(I\). So \(\exists x : P(x) \lor \exists x : Q(x)\) is true for \(I\) and \(I\) is a model for \(\exists x : P(x) \lor \exists x : Q(x)\).

If \(I\) is a model for \(\exists x : P(x) \lor \exists x : Q(x)\) with domain \(D\), then \(\exists x : P(x) \lor \exists x : Q(x)\) is true for \(I\), which means that at least one of \(\exists x : P(x)\) and \(\exists x : Q(x)\) is true for \(I\), which means that there is some \(d \in D\) such that at least one of \(P(d)\) and \(Q(d)\) is true. So there is some \(d \in D\) such that \(P(d) \lor Q(d)\) is true. So \(\exists x : P(x) \lor Q(x)\) is true for \(I\), and \(I\) is a model for \(\exists x : P(x) \lor Q(x)\).

6. The statement is false. Consider the interpretation where the domain is \(\{true, false\}\), \(P(x)\) is \(x\) and \(Q(x)\) is \(\neg x\). Then the statement becomes:

\(\exists x : (x \land \neg x) \equiv \exists x : x \land \exists x : \neg x\)

The left hand side of the statement is false, while the right hand side is true.

7. To negate an expression with \(n\) quantifiers, we can replace each \(\exists\) in the sequence with \(\forall\), each \(\forall\) in the sequence with \(\exists\), and negate the predicate inside the sequence.

We will prove this statement by induction on the number of quantifiers in the sequence.

Base case: there is a single quantifier, and the statement is therefore true (proved in class).

Induction step: assume the statement is true for \(n\). We now have to prove it for \(n + 1\).

Let \(\otimes\) represent either \(\exists\) or \(\forall\), and \(\otimes\) represent \(\otimes\) replaced with the other quantifier. Then:

\(\neg((\otimes x_1 \ldots \otimes x_{n+1} : P(x_1, \ldots, x_{n+1})) \equiv \neg((\otimes x_1 : Q(x_1)) \equiv \otimes x_1 : \neg Q(x_1))\)

But \(Q(x_1)\) is an expression with \(n\) quantifiers, so by the induction hypothesis, \(\neg Q(x_1) \equiv \otimes x_2 \ldots \otimes x_{n+1} : \neg P(x_1, \ldots, x_{n+1})\). So:

\(\neg((\otimes x_1 \ldots \otimes x_{n+1} : P(x_1, \ldots, x_{n+1})) \equiv \otimes x_1 \otimes x_2 \ldots \otimes x_{n+1} : \neg P(x_1, \ldots, x_{n+1})\)

which proves the statement to be true for \(n + 1\), which completes the proof.