Simplifications of Context-Free Grammars

A Substitution Rule

$$S \to aB$$
$$A \to aaA$$
$$A \to abBc$$
$$B \to aA$$
$$B \to b$$

Substitute

$$B \to b$$

Equivalent grammar

$$S \to aB \mid ab$$
$$A \to aaA$$
$$A \to abBc \mid abbc$$
$$B \to aA$$

A Substitution Rule

$$S \to aB \mid ab$$

$$A \to aaA$$

$$A \to abBc \mid abbc$$

$$B \to aA$$

Substitute

$$B \to aA$$

$$S \to \cancel{aB} \mid ab \mid aaA$$

$$A \to aaA$$

$$A \to \cancel{abBc} \mid abbc \mid abaAc$$

Equivalent grammar

In general:

$$A \to xBz$$

$$B \to y_1$$

Substitute

$$B \to y_1$$

$$A \to xBz \mid xy_1z$$

equivalent grammar

Nullable Variables

$\lambda - \text{production}:$ $A \rightarrow \lambda$

Nullable Variable: $A \Rightarrow \ldots \Rightarrow \lambda$

Removing Nullable Variables

Example Grammar:

$$S \to aMb$$

$$M \to aMb$$

$$M \to \lambda$$

Nullable variable

$$S \to aMb$$

$$M \to aMb$$

$$\cancel{M \to \lambda}$$

Substitute

$$M \to \lambda$$

Final Grammar

$$S \to aMb$$

$$S \to ab$$

$$M \to aMb$$

$$M \to ab$$

Unit-Productions

Unit Production: $A \to B$

(a single variable in both sides)

Removing Unit Productions

Observation:

$$A \to A$$

Is removed immediately

Example Grammar:

$$S \to aA$$

$$A \to a$$

$$A \to B$$

$$B \to A$$

$$B \to bb$$

$$S \to aA$$

$$A \to a$$

$$\cancel{A \to B}$$

$$B \to A$$

$$B \to bb$$

Substitute

$$A \to B$$

$$S \to aA \mid aB$$

$$A \to a$$

$$B \to A \mid B$$

$$B \to bb$$

$$S \to aA \mid aB$$

$$A \to a$$

$$B \to A \mid \cancel{B}$$

$$B \to bb$$

Remove

$$B \to B$$

$$S \to aA \mid aB$$

$$A \to a$$

$$B \to A$$

$$B \to bb$$

$$S \to aA \mid aB$$

$$A \to a$$

$$\cancel{B \to A}$$

$$B \to bb$$

Substitute

$$B \to A$$

$$S \to aA \mid aB \mid aA$$

$$A \to a$$

$$B \to bb$$

Remove repeated productions

$$S \rightarrow aA \mid aB \mid \cancel{aA}$$

$$A \rightarrow a$$

$$B \rightarrow bb$$

Final grammar

$$S \rightarrow aA \mid aB$$

$$A \rightarrow a$$

$$B \rightarrow bb$$

Useless Productions

$$S \to aSb$$

$$S \to \lambda$$

$$S \to A$$

$$A \to aA$$ Useless Production

Some derivations never terminate...

$$S \Rightarrow A \Rightarrow aA \Rightarrow aaA \Rightarrow \ldots \Rightarrow aa\ldots aA \Rightarrow \ldots$$

Another grammar:

$$S \to A$$

$$A \to aA$$

$$A \to \lambda$$

$$B \to bA$$

Useless Production

Not reachable from S

In general:

contains only terminals

if $S \Rightarrow \ldots \Rightarrow xAy \Rightarrow \ldots \Rightarrow w$

$w \in L(G)$

then variable A is useful

otherwise, variable A is useless

A production $A \to x$ is useless
if any of its variables is useless

Productions

Variables

$S \to aSb$

$S \to \lambda$

$S \to A$ useless

useless $A \to aA$ useless

useless $B \to C$ useless

useless $C \to D$ useless

Removing Useless Productions

Example Grammar:

$$S \to aS \mid A \mid C$$

$$A \to a$$

$$B \to aa$$

$$C \to aCb$$

First: find all variables that can produce strings with only terminals

$$S \to aS \mid A \mid C$$

$$A \to a$$

$$B \to aa$$

$$C \to aCb$$

Round 1: $\{A, B\}$

$$S \to A$$

Round 2: $\{A, B, S\}$

Keep only the variables
that produce terminal symbols: $\{A, B, S\}$

(the rest variables are useless)

$$S \rightarrow aS \mid A \mid \cancel{C}$$

$$A \rightarrow a$$

$$B \rightarrow aa$$

$$\cancel{C \rightarrow aCb}$$

$$\Rightarrow$$

$$S \rightarrow aS \mid A$$

$$A \rightarrow a$$

$$B \rightarrow aa$$

Remove useless productions

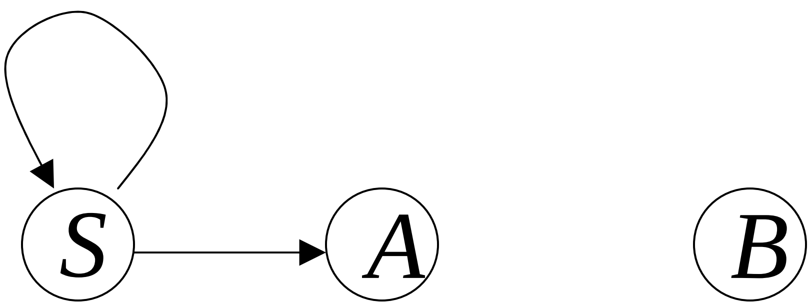
Second: Find all variables reachable from S

$$S \to aS \mid A$$

$$A \to a$$

$$B \to aa$$

Use a Dependency Graph



not reachable

Keep only the variables reachable from S

(the rest variables are useless)

Final Grammar

$$S \to aS \mid A$$

$$A \to a$$

$$\cancel{B \to aa}$$

$$\Rightarrow$$

$$S \to aS \mid A$$

$$A \to a$$

Remove useless productions

Removing All

Step 1: Remove λ-productions

Step 2: Remove Unit-productions

Step 3: Remove Useless productions

Normal Forms for Context-free Grammars

Chomsky Normal Form

Each productions has form:

$$A \to BC \quad \text{or} \quad A \to a$$

variable variable terminal

Examples:

$$S \to AS$$

$$S \to a$$

$$A \to SA$$

$$A \to b$$

Chomsky Normal Form

$$S \to AS$$

$$S \to AAS$$

$$A \to SA$$

$$A \to aa$$

Not Chomsky Normal Form

Conversion to Chomsky Normal Form

Example:

$$S \to ABa$$

$$A \to aab$$

$$B \to Ac$$

Not Chomsky Normal Form

Introduce variables for terminals: T_a, T_b, T_c

$$S \to ABa$$
$$A \to aab$$
$$B \to Ac$$

$$S \to ABT_a$$
$$A \to T_a T_a T_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Introduce intermediate variable: V_1

$$S \to ABT_a$$
$$A \to T_aT_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$A \to T_aT_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Introduce intermediate variable: V_2

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$A \to T_aT_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

$\Longrightarrow$

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$A \to T_aV_2$$
$$V_2 \to T_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Final grammar in Chomsky Normal Form:

Initial grammar

$$S \to ABa$$

$$A \to aab$$

$$B \to Ac$$

$$S \to AV_1$$

$$V_1 \to BT_a$$

$$A \to T_a V_2$$

$$V_2 \to T_a T_b$$

$$B \to AT_c$$

$$T_a \to a$$

$$T_b \to b$$

$$T_c \to c$$

In general:

From any context-free grammar
(which doesn't produce λ)
not in Chomsky Normal Form

we can obtain:

An equivalent grammar
in Chomsky Normal Form

The Procedure

First remove:

Nullable variables

Unit productions

Then, for every symbol a:

Add production $T_a \to a$

In productions: replace a with T_a

New variable: T_a

Replace any production $A \to C_1 C_2 \cdots C_n$

with
$$A \to C_1 V_1$$
$$V_1 \to C_2 V_2$$
$$\cdots$$
$$V_{n-2} \to C_{n-1} C_n$$

New intermediate variables: $V_1, V_2, \ldots, V_{n-2}$

Theorem: For any context-free grammar (which doesn't produce λ) there is an equivalent grammar in Chomsky Normal Form

Observations

- Chomsky normal forms are good for parsing and proving theorems

- It is very easy to find the Chomsky normal form for any context-free grammar

Greibach Normal Form

All productions have form:

$$A \to a V_1 V_2 \cdots V_k \qquad k \geq 0$$

symbol

variables

Examples:

$$S \to cAB$$

$$A \to aA \mid bB \mid b$$

$$B \to b$$

Greibach Normal Form

$$S \to abSb$$

$$S \to aa$$

Not Greibach Normal Form

Conversion to Greibach Normal Form:

$$S \to abSb$$

$$S \to aa$$

$$S \to aT_bST_b$$

$$S \to aT_a$$

$$T_a \to a$$

$$T_b \to b$$

Greibach
Normal Form

Theorem: For any context-free grammar (which doesn't produce λ) there is an equivalent grammar in Greibach Normal Form

Observations

- Greibach normal forms are very good for parsing

- It is hard to find the Greibach normal form of any context-free grammar

The CYK Parser

The CYK Membership Algorithm

Input:

- Grammar G in Chomsky Normal Form

- String w

Output:

find if $w \in L(G)$

The Algorithm

Input example:

- Grammar G :

$$S \to AB$$

$$A \to BB$$

$$A \to a$$

$$B \to AB$$

$$B \to b$$

- String w : $aabbb$

aabbb

a	a	b	b	b
aa	ab	bb	bb	
aab	abb	bbb		
aabb	abbb			
aabbb				

$$S \to AB$$

$$A \to BB$$

$$A \to a$$

$$B \to AB$$

$$B \to b$$

a	a	b	b	b
A	A	B	B	B
aa	ab	bb	bb	
aab	abb	bbb		
aabb	abbb			
aabbb				

$$S \to AB$$

$$A \to BB$$

$$A \to a$$

$$B \to AB$$

$$B \to b$$

a	a	b	b	b
A	A	B	B	B
aa	ab	bb	bb	
	S,B	A	A	
aab	abb	bbb		
aabb	abbb			
aabbb				

$$S \rightarrow AB$$

$$A \rightarrow BB$$

$$A \rightarrow a$$

$$B \rightarrow AB$$

$$B \rightarrow b$$

a A	a A	b B	b B	b B
aa	ab S,B	bb A	bb A	
aab S,B	abb A	bbb S,B		
aabb A	abbb S,B			
aabbb S,B				

Therefore: $aabbb \in L(G)$

Time Complexity: $|w|^3$

Observation: The CYK algorithm can be easily converted to a parser (bottom up parser)