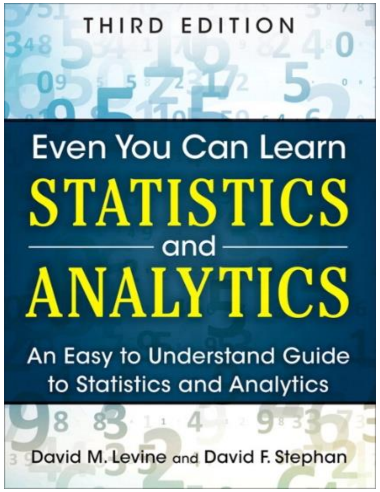
IMGD 2905

Simple Linear Regression

Chapter 10



Motivation

- Have data (sample, **x**’s. e.g., *playtime*)
- Want to know likely value of next observation (A)

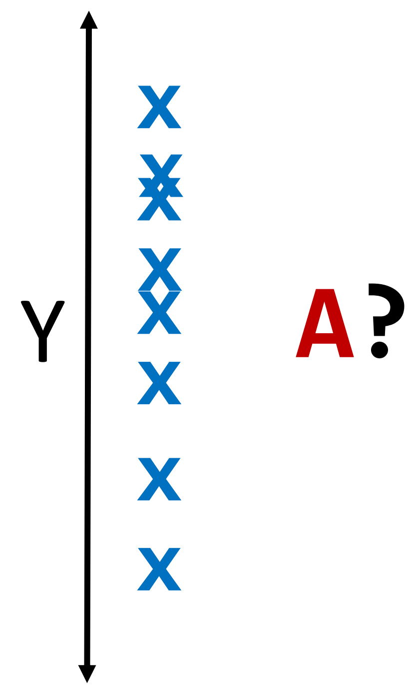
A. Compute mean y-value (with confidence interval)

→ Predict A

- But what if have additional information?

  E.g., *playtime* versus *skins owned*

→ Better prediction!

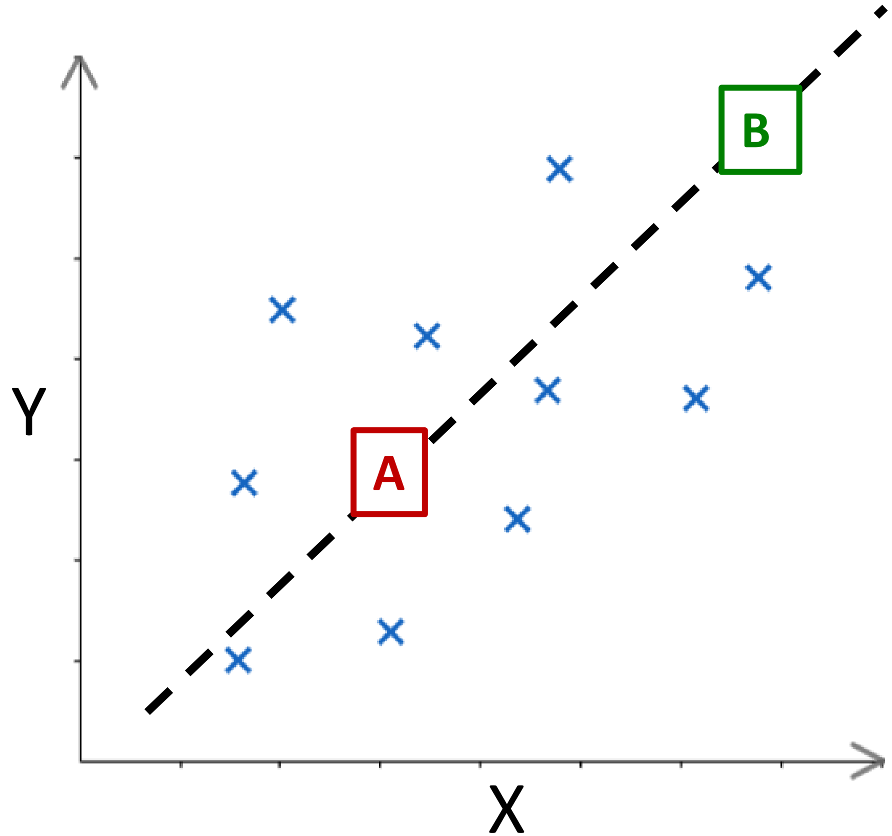


Motivation

- Have data (sample, **x**'s), based on X
  E.g., *playtime* versus *skins owned*
- Want to know likely value of next observation (Y)
- A – reasonable to compute mean y-value (with confidence interval)
- B – could do same, but there appears to be relationship between X and Y!

→ Predict B (here, use X data to predict Y)

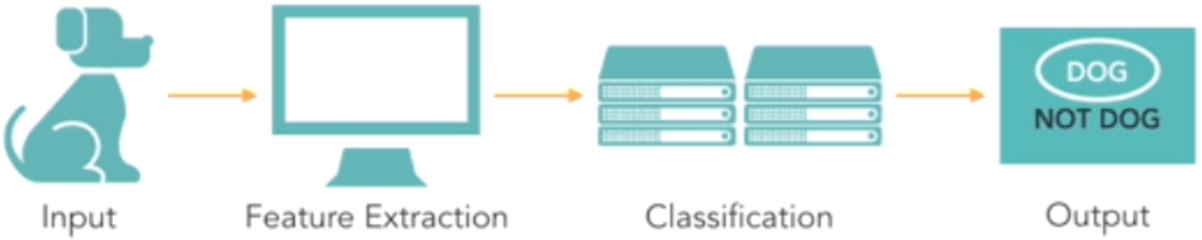
e.g., "trendline" (regression)



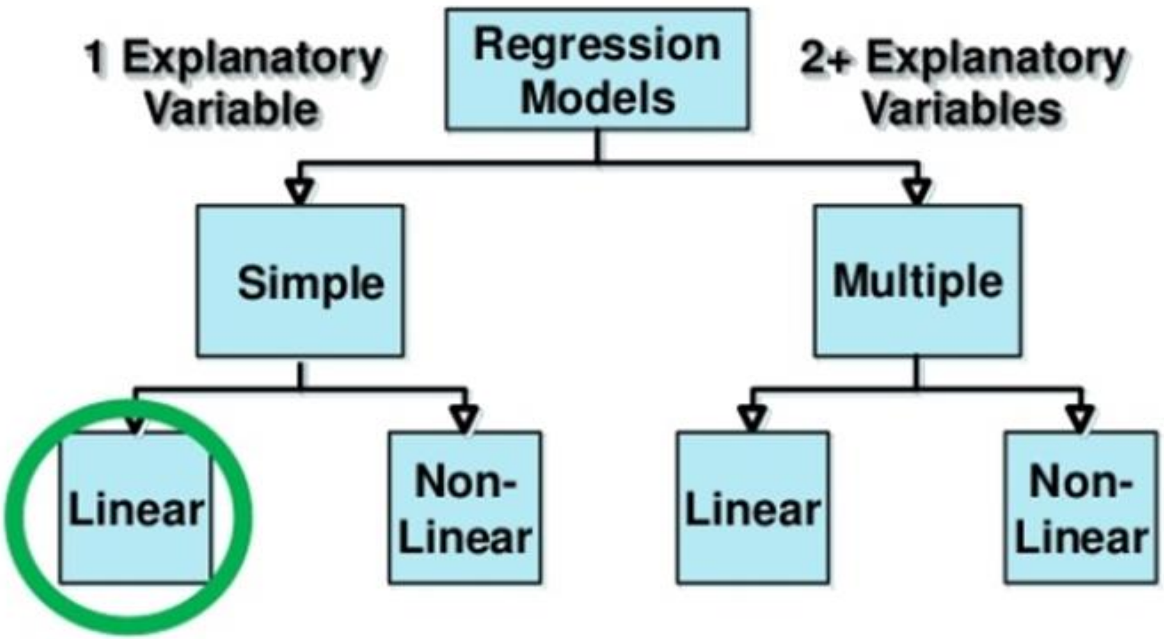
Overview

Broadly, two types of prediction techniques:

1. Regression – mathematical equation to model, then use model for predictions
   – We'll discuss simple linear regression
2. Machine learning – branch of AI, use computer algorithms to determine relationships (predictions)
   – CS 4342 Machine Learning



Types of Regression Models



- Explanatory variable *explains* dependent variable
  - Variable **X** (e.g., skill level) explains **Y** (e.g., KDA)
  - Can have **1 (simple)** or 2+ (multiple)
- **Linear** if coefficients added, else Non-linear

Outline

- Introduction (done)
- Simple Linear Regression (next)
  - Linear relationship
  - Residual analysis
  - Fitting parameters
- Measures of Variation
- Misc

Simple Linear Regression

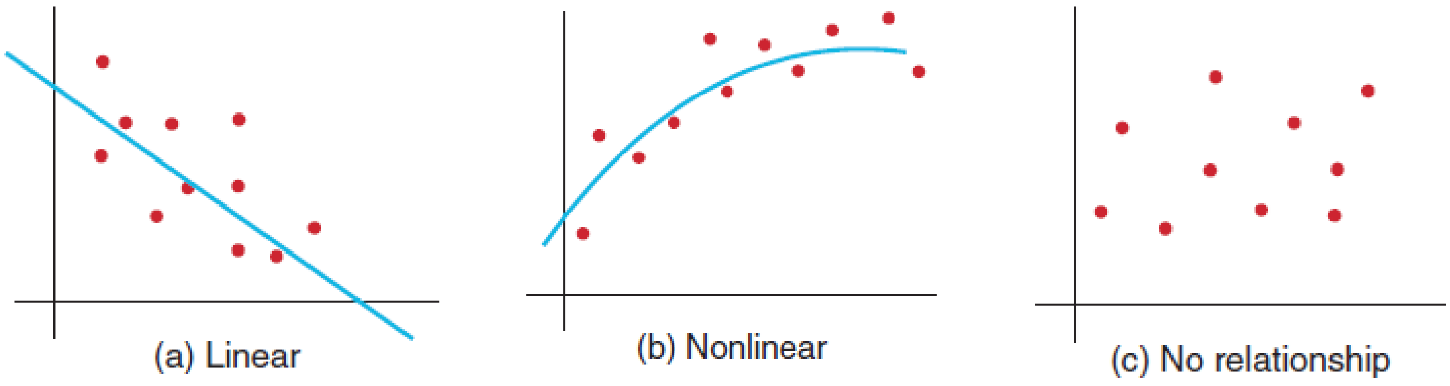
- Goal – find a linear (line) relationship between two values
  - E.g., *travel time* and *car speed, KDA* and *skill,*
- First, make sure relationship is linear! How?

→ Scatterplot

(c) no clear relationship

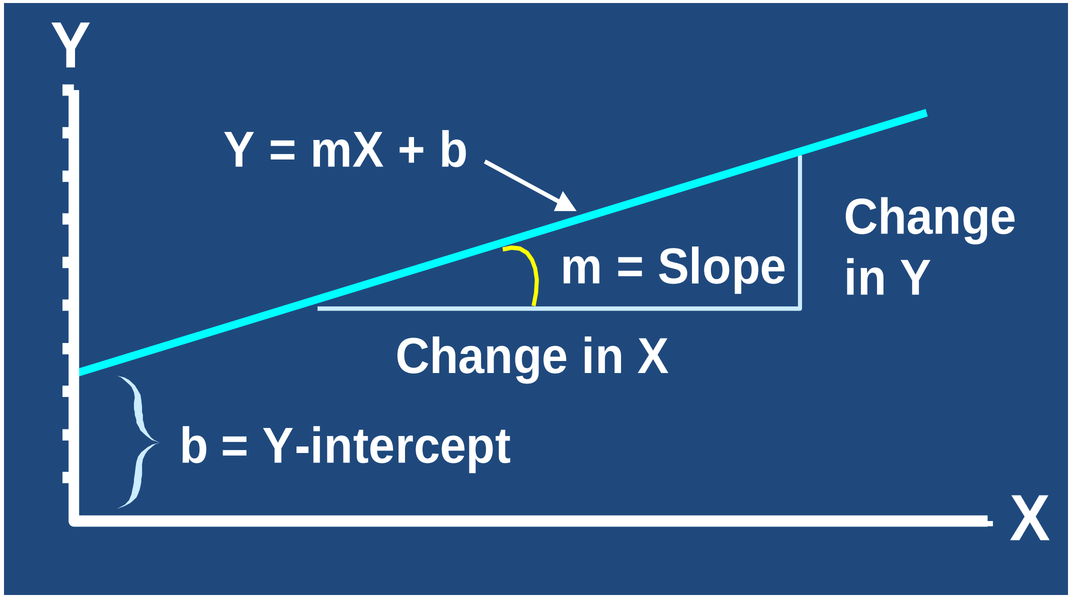
(b) not a linear relationship

(a) linear relationship – proceed with linear regression



Linear Relationship

- From algebra: line in form $Y = mX + b$
  - m is slope, b is y-intercept
- Slope (m) is amount *Y* increases when *X* increases by 1 unit (specifying units important!)
- Intercept (b) is where line crosses y-axis, or where y-value when *x* = 0



https://www.scribd.com/presentation/230686725/Fu-Ch11-Linear-Regression

Simple Linear Regression Example

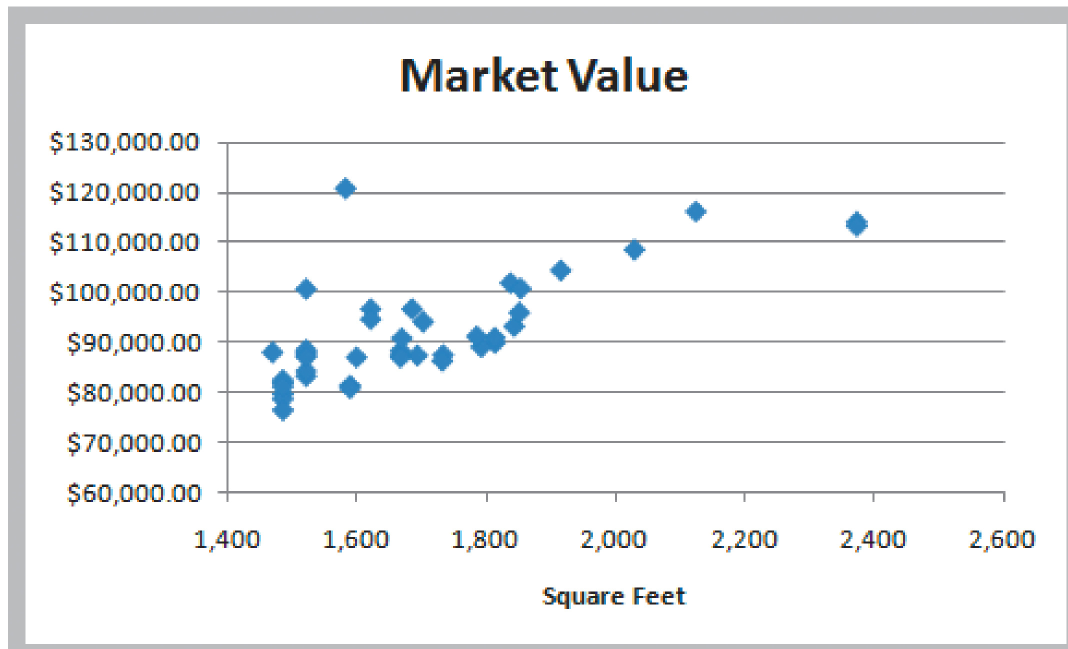
- Market value of house is related to size

  X = square footage

  Y = market value ($)

- Scatter plot (42 homes)
  - indicates linear trend

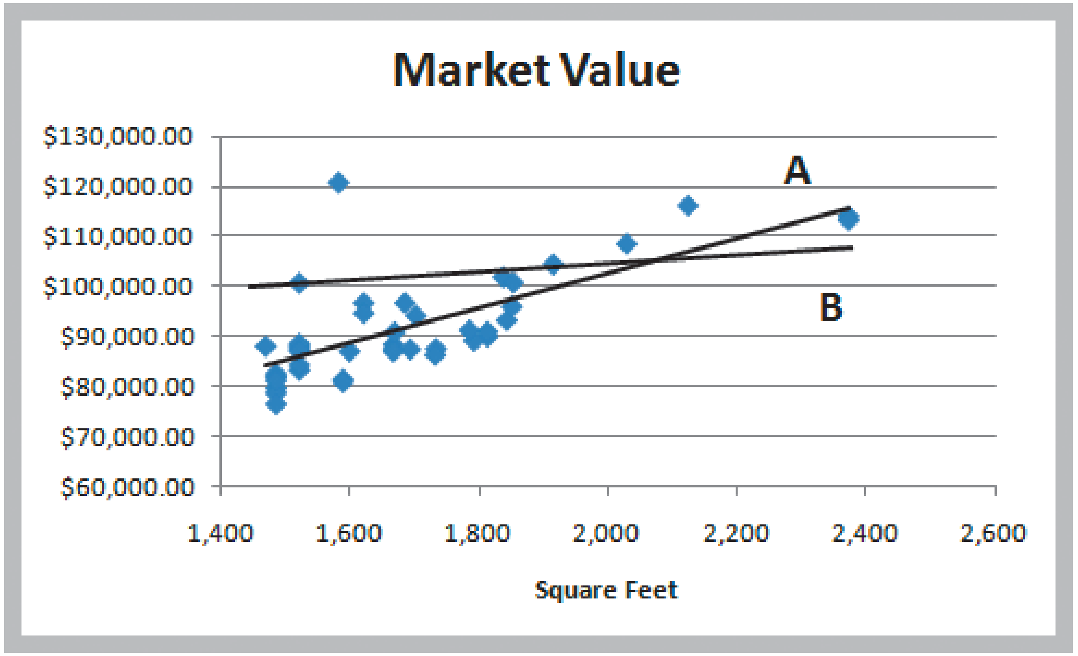
	A	B	C
1	Home Market Value		
2			
3	House Age	Square Feet	Market Value
4	33	1,812	$90,000.00
5	32	1,914	$104,400.00
6	32	1,842	$93,300.00
7	33	1,812	$91,000.00
8	32	1,836	$101,900.00
9	33	2,028	$108,500.00
10	32	1,732	$87,600.00



Simple Linear Regression Example

- Two possible lines shown below (A and B)
- Want to determine best regression line
- Line A looks a better fit to data
  - But how to know?

$Y = mX + b$

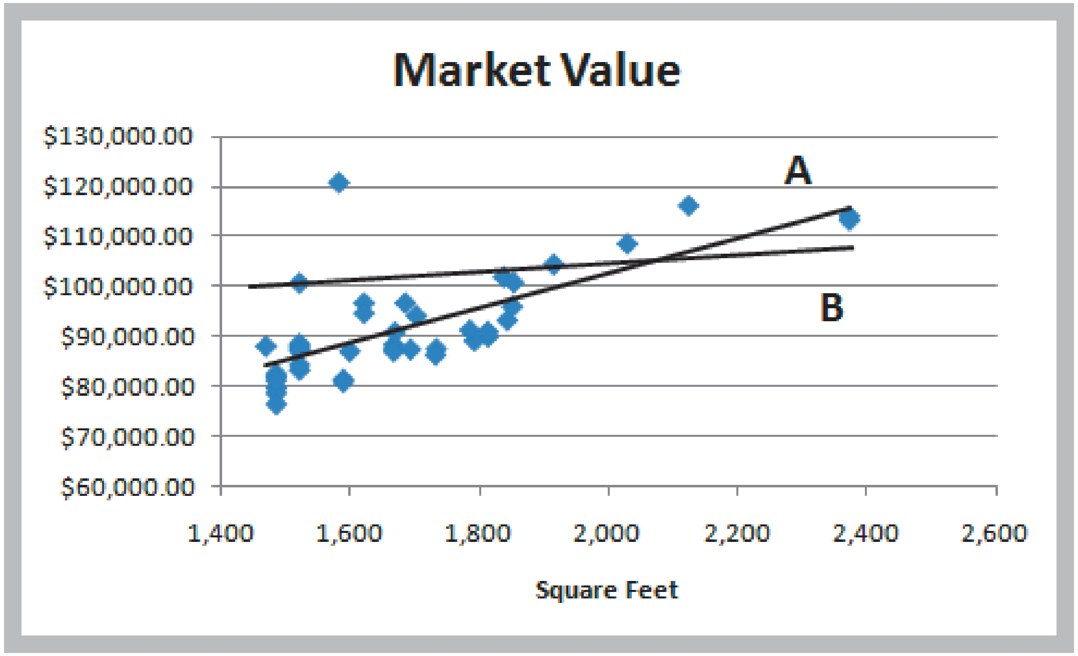


Simple Linear Regression Example

- Two possible lines shown below (A and B)
- Want to determine best regression line
- Line A looks a better fit to data
  - But how to know?

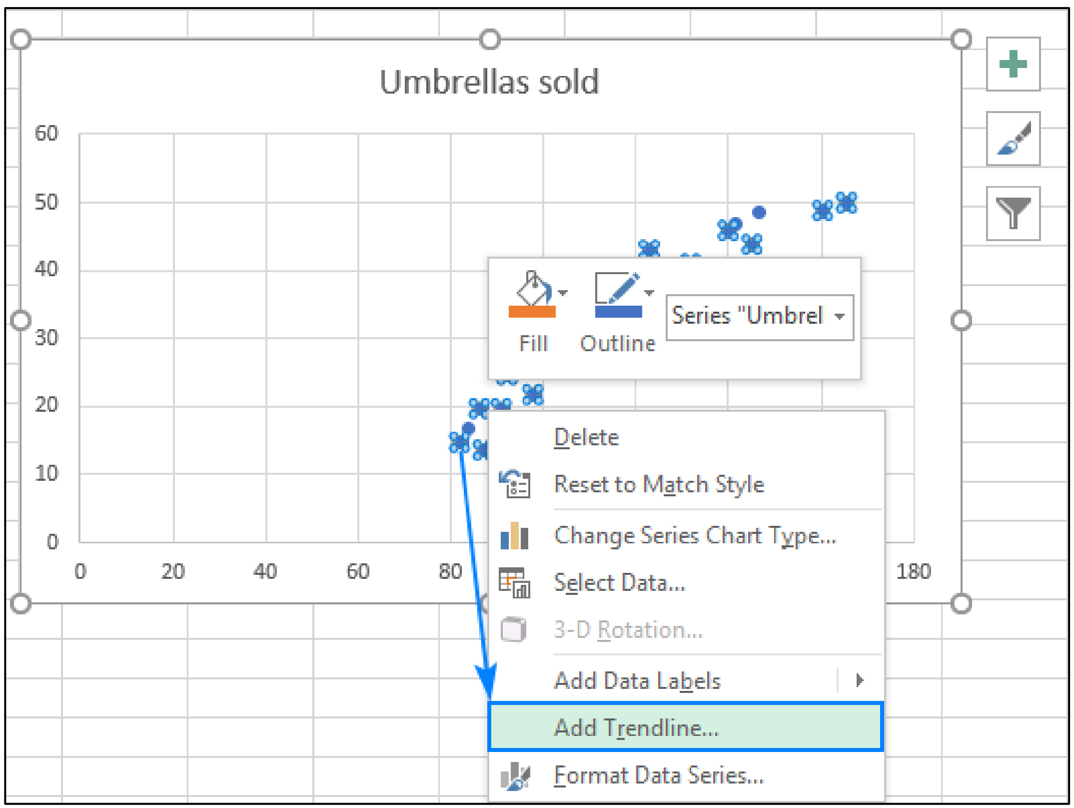
$Y = mX + b$

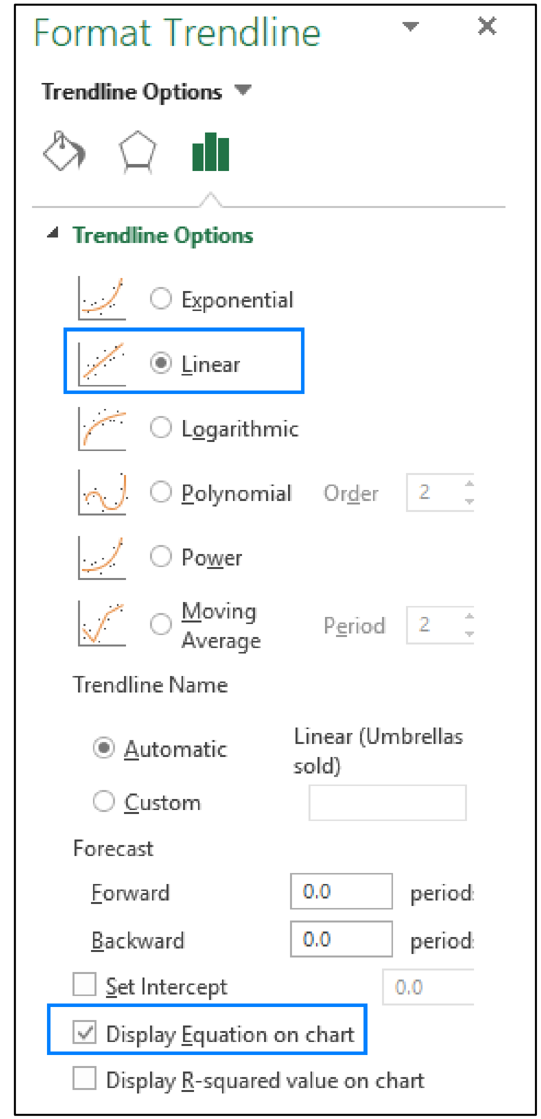
Line that gives best fit to data is one that minimizes prediction error
→ Least squares line
(more later)



Simple Linear Regression Example Chart

- Scatterplot
- Right click → Add Trendline





Simple Linear Regression Example
Formulas

`=SLOPE(C4:C45,B4:B45)`

→ Slope = 35.04

`=INTERCEPT(C4:C45,B4:B45)`

→ Intercept = 32,600

	A	B	C
1	**Home Market Value**		
2			
3	**House Age**	**Square Feet**	**Market Value**
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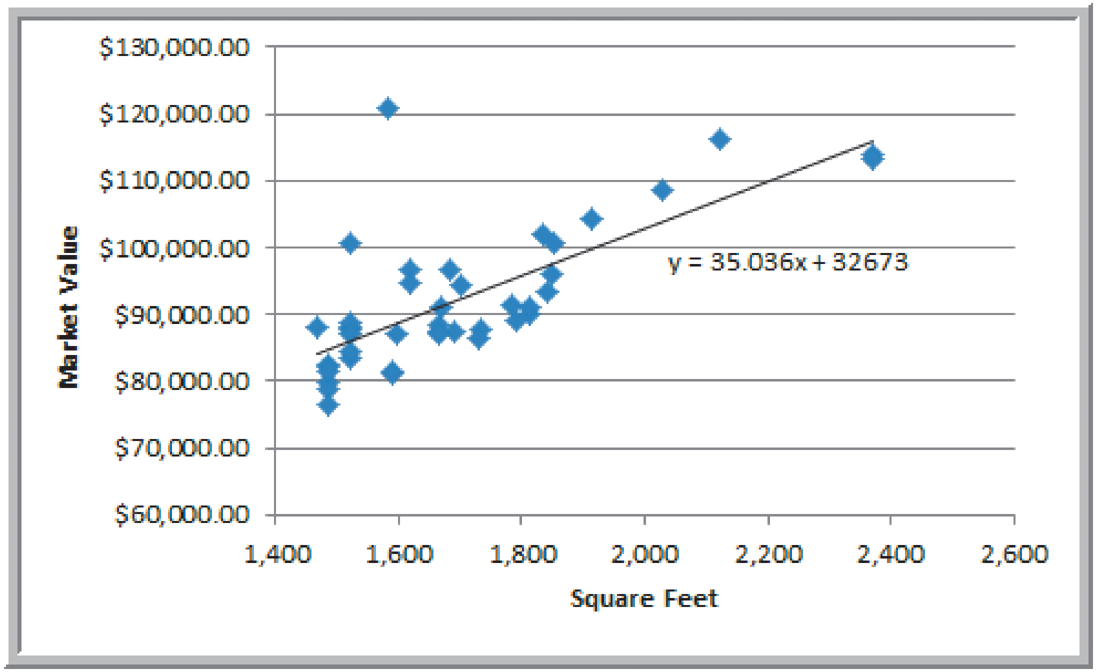
Estimate Y when X = 1800 square feet

Y = 32600 + 35.04 x (1800) = $95,672

Simple Linear Regression Example

Market value = 32600 + 35.04 *x* (square feet)

Predicts market value better than just average



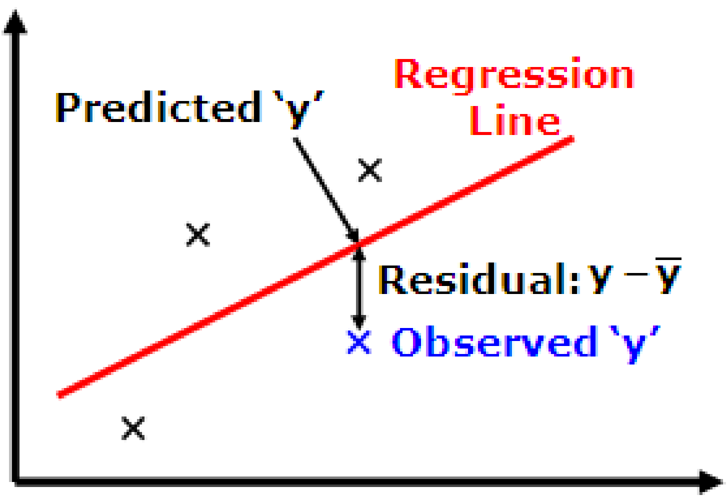
But before use, examine residuals

Outline

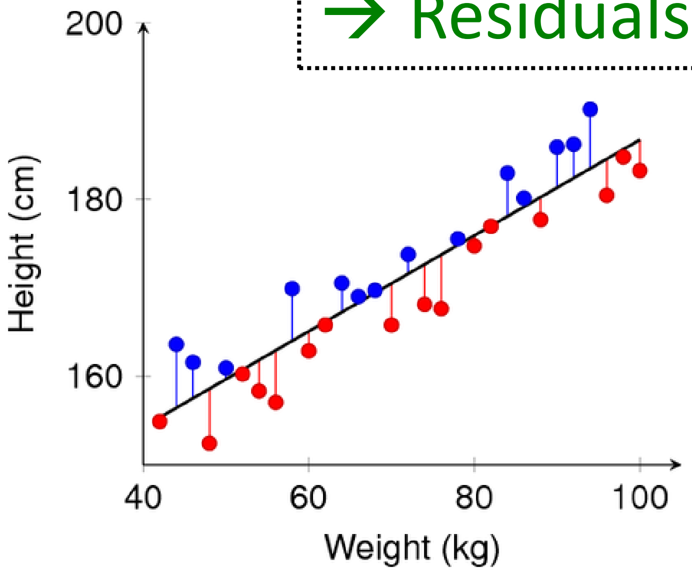
- Introduction (done)
- Simple Linear Regression
  - Linear relationship (done)
  - Residual analysis (next)
  - Fitting parameters
- Measures of Variation
- Misc

Residual Analysis

- Before predicting, confirm that linear regression assumptions hold
  - Variation around line is normally distributed
  - Variation equal for all *X*
  - Variation independent for all *X*
- How? Compute residuals (error in prediction)

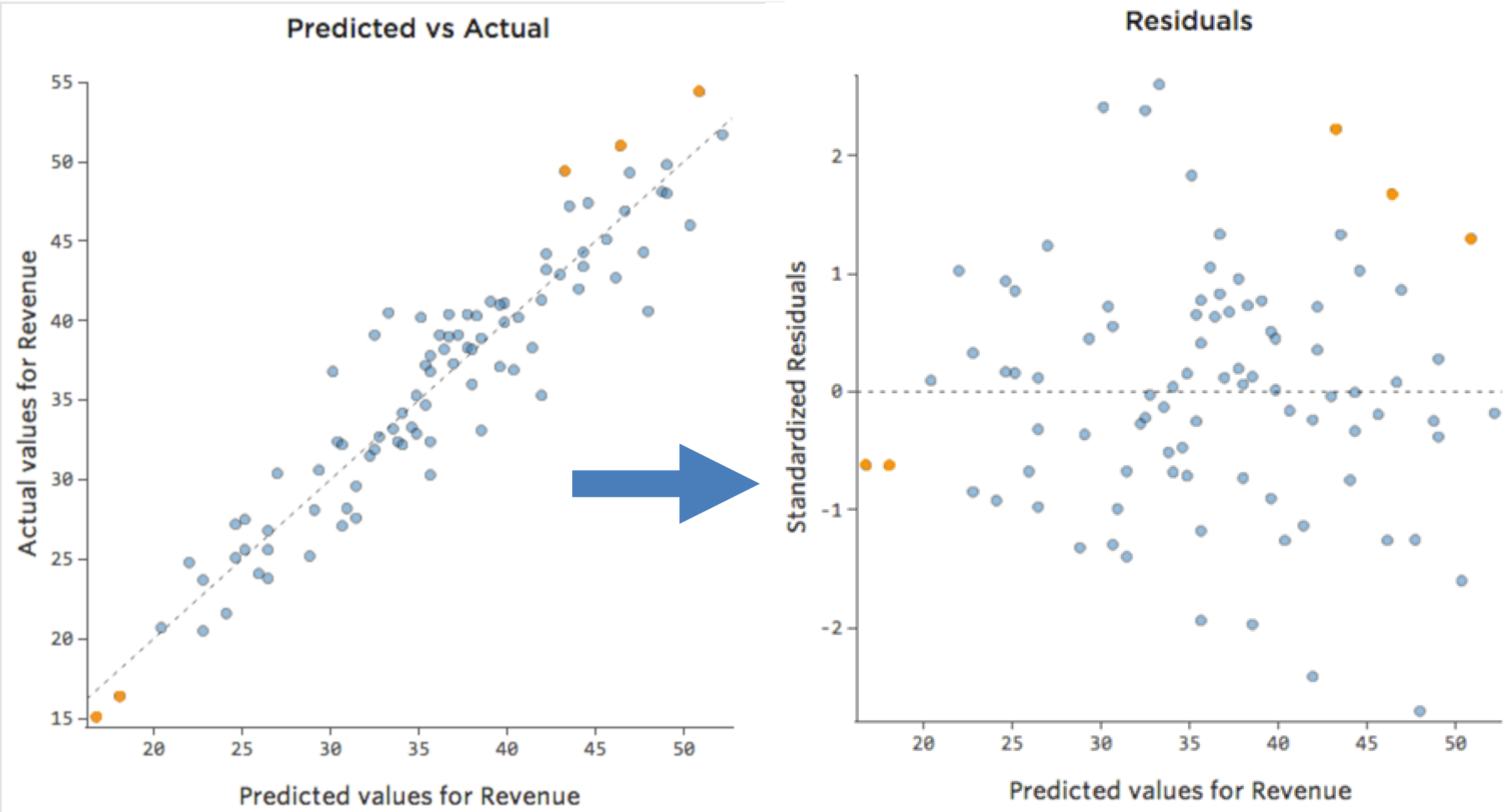


→ Residuals Chart



Residual Analysis

https://www.qualtrics.com/support/stats-iq/analyses/regression-guides/interpreting-residual-plots-improve-regression/



Note that we've colored in a few dots in orange so you can get the sense of how this transformation works.

Variation around line normally distributed ?
Variation equal for all X
Variation independent for all X?

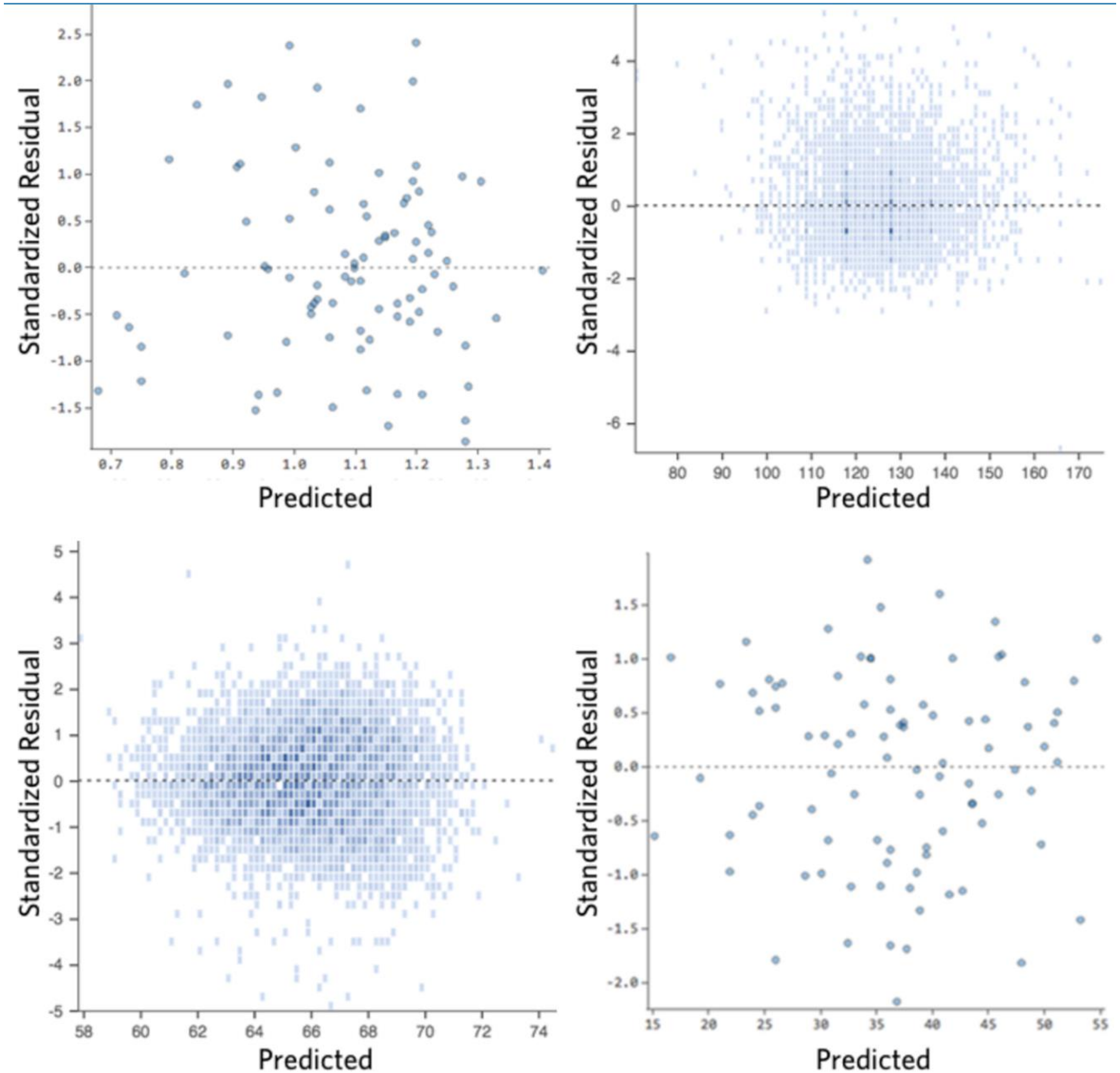
No clear pattern

Residual Analysis – Good

https://www.qualtrics.com/support/stats-iq/analyses/regression-guides/interpreting-residual-plots-improve-regression/

Symmetrically distributed

No clear pattern



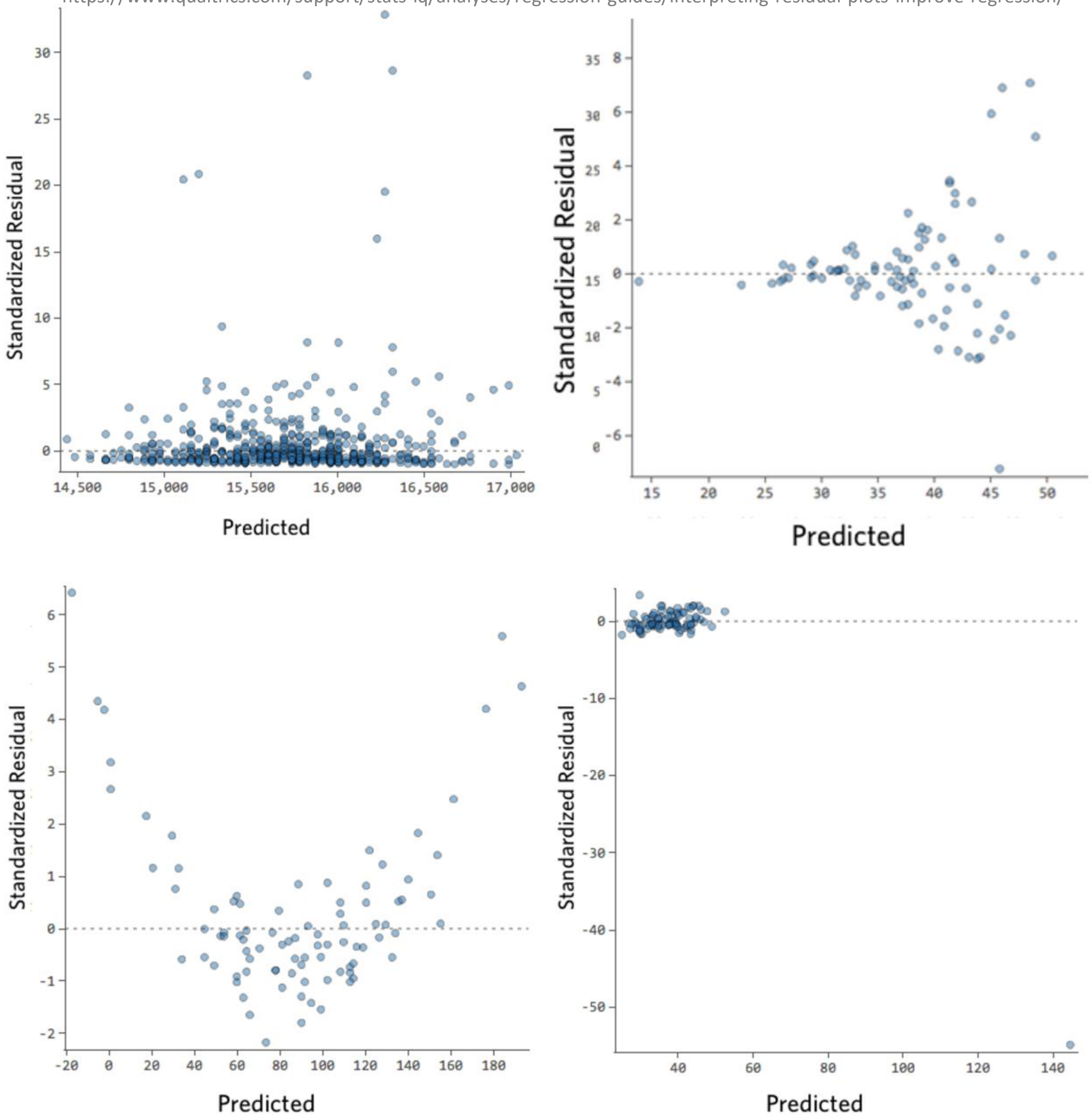
Clustered towards middle, ok since need normally distributed

Residual Analysis – Bad

https://www.qualtrics.com/support/stats-iq/analyses/regression-guides/interpreting-residual-plots-improve-regression/

Clear shape

Outliers



Patterns!

Note: could do normality test (QQ plot)

Outline

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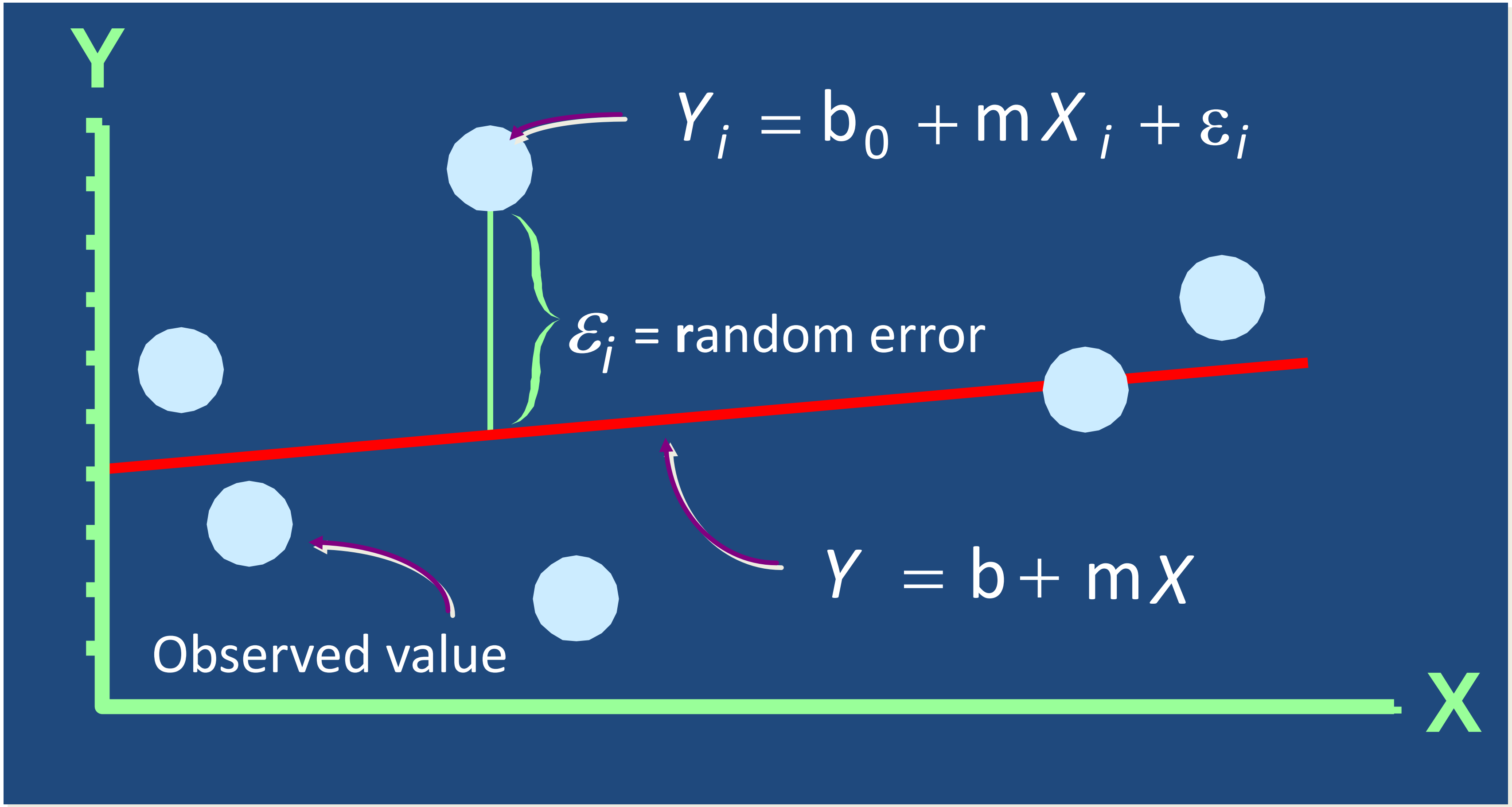
Linear Regression Model

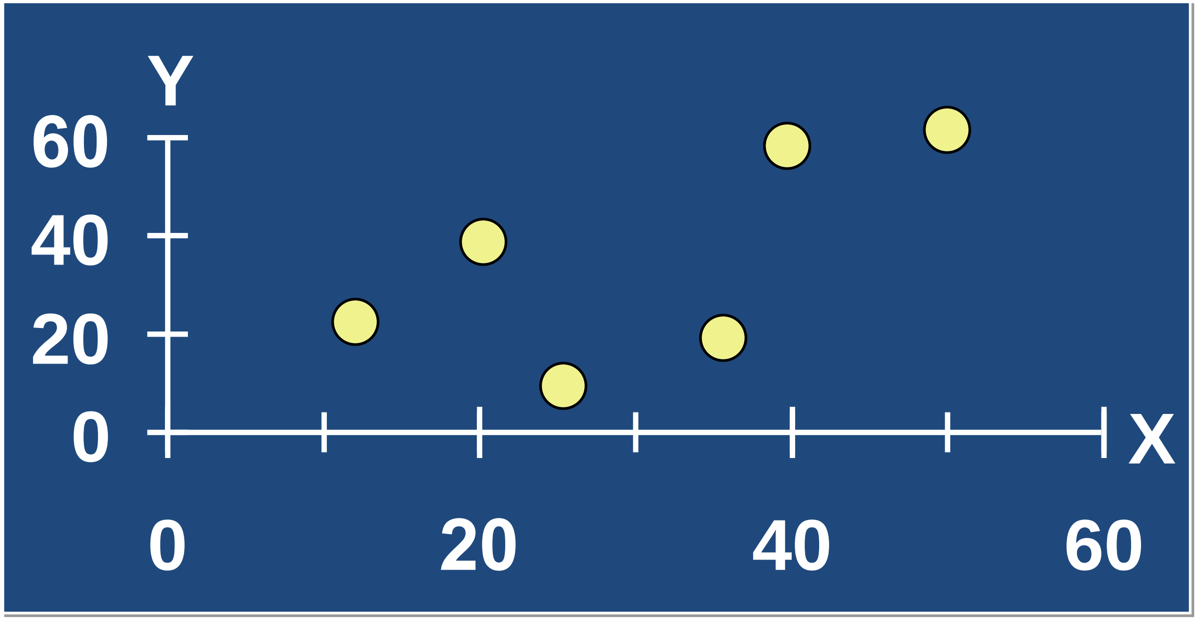


https://www.scribd.com/presentation/230686725/Fu-Ch11-Linear-Regression

Random error associated with each observation (Residual)

Fitting the Best Line

- Plot all (X_i, Y_i) Pairs



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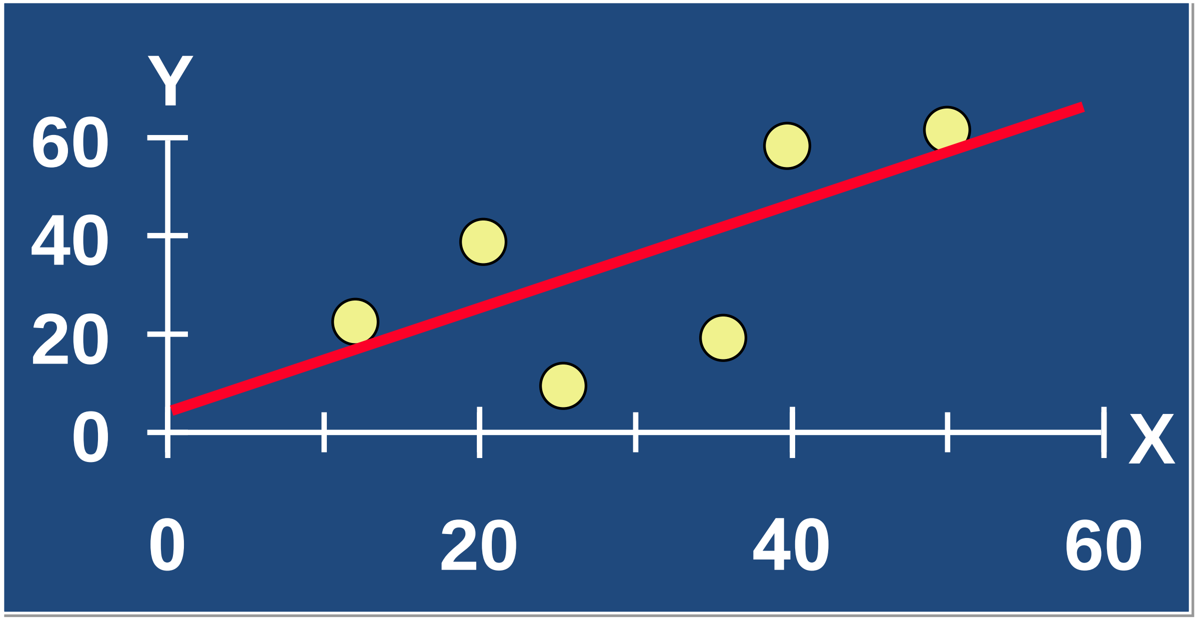
Fitting the Best Line

- Plot all (X_i, Y_i) Pairs
- Draw a line. But how do we know it is best?



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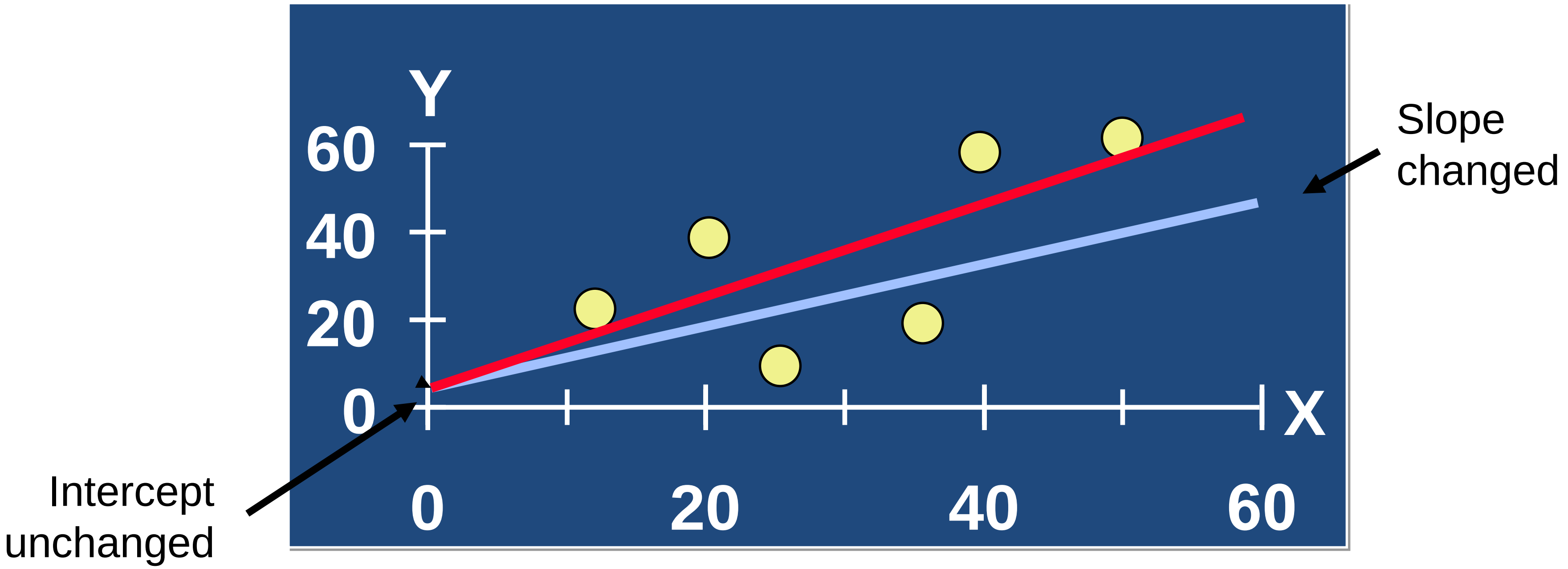

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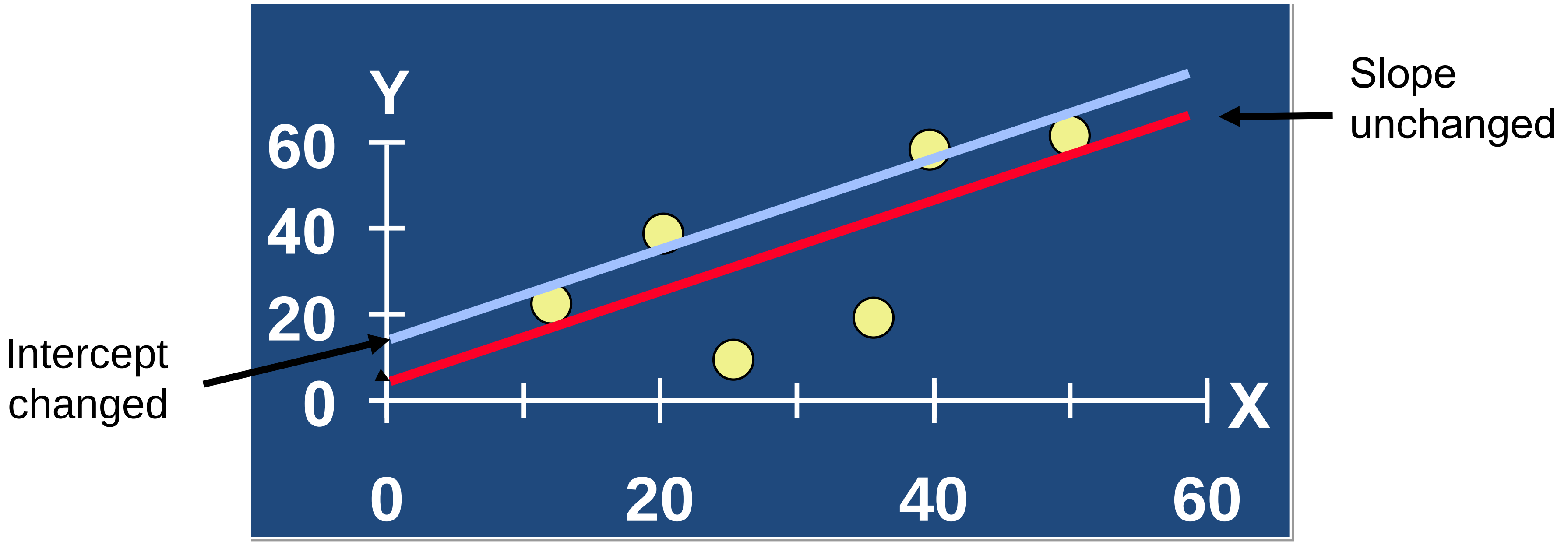

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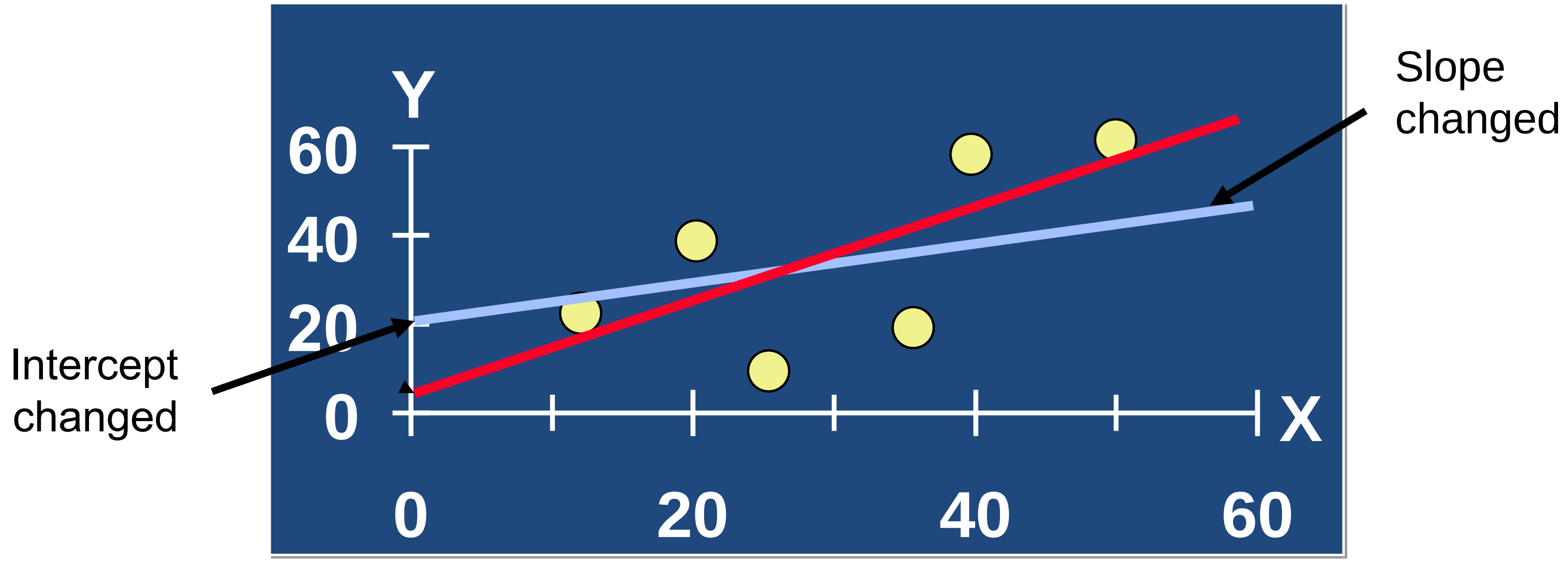

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Linear Regression Model

- Relationship between variables is linear function

Population Y-Intercept

Population Slope

Random Prediction Error

$$Y_i = b_0 + m X_i + \varepsilon_i$$

Want error as small as possible

Dependent (response) Variable (e.g., kills)

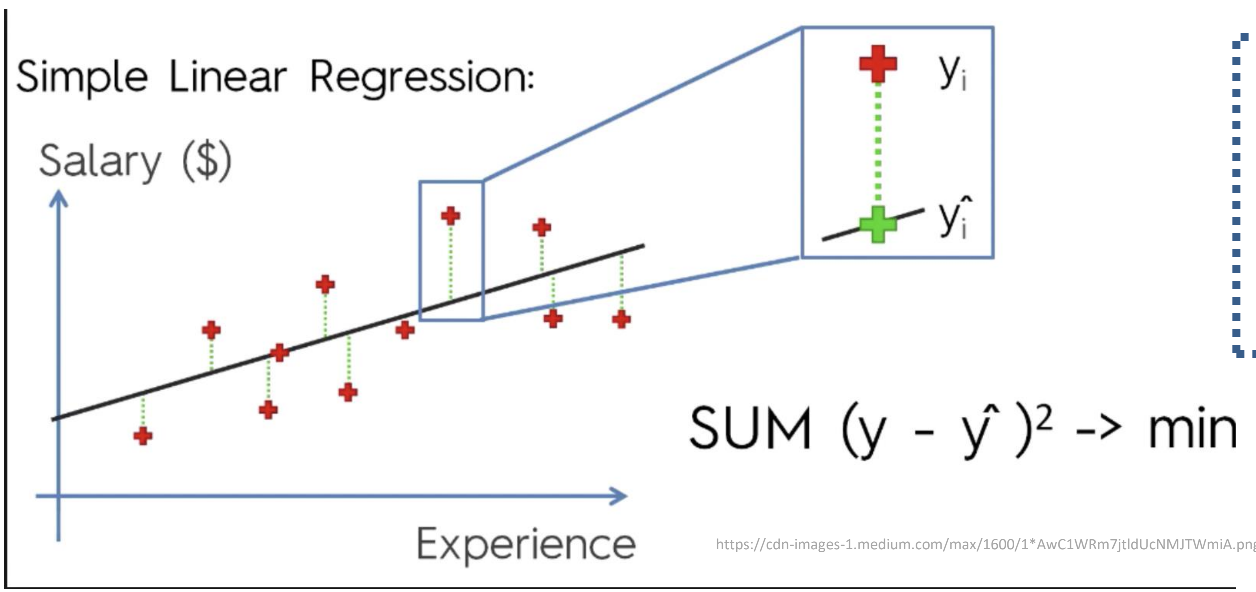
i-th observation / prediction

Independent (explanatory) Variable (e.g., skill level)

Least Squares Line

- Want to minimize difference between actual y and predicted ŷ
  - Add up ε_i for all observed y's
  - But positive differences offset negative ones!
  - (remember when this happened for variance?)

  → Square the errors! Then, minimize (using Calculus)



Minimize:

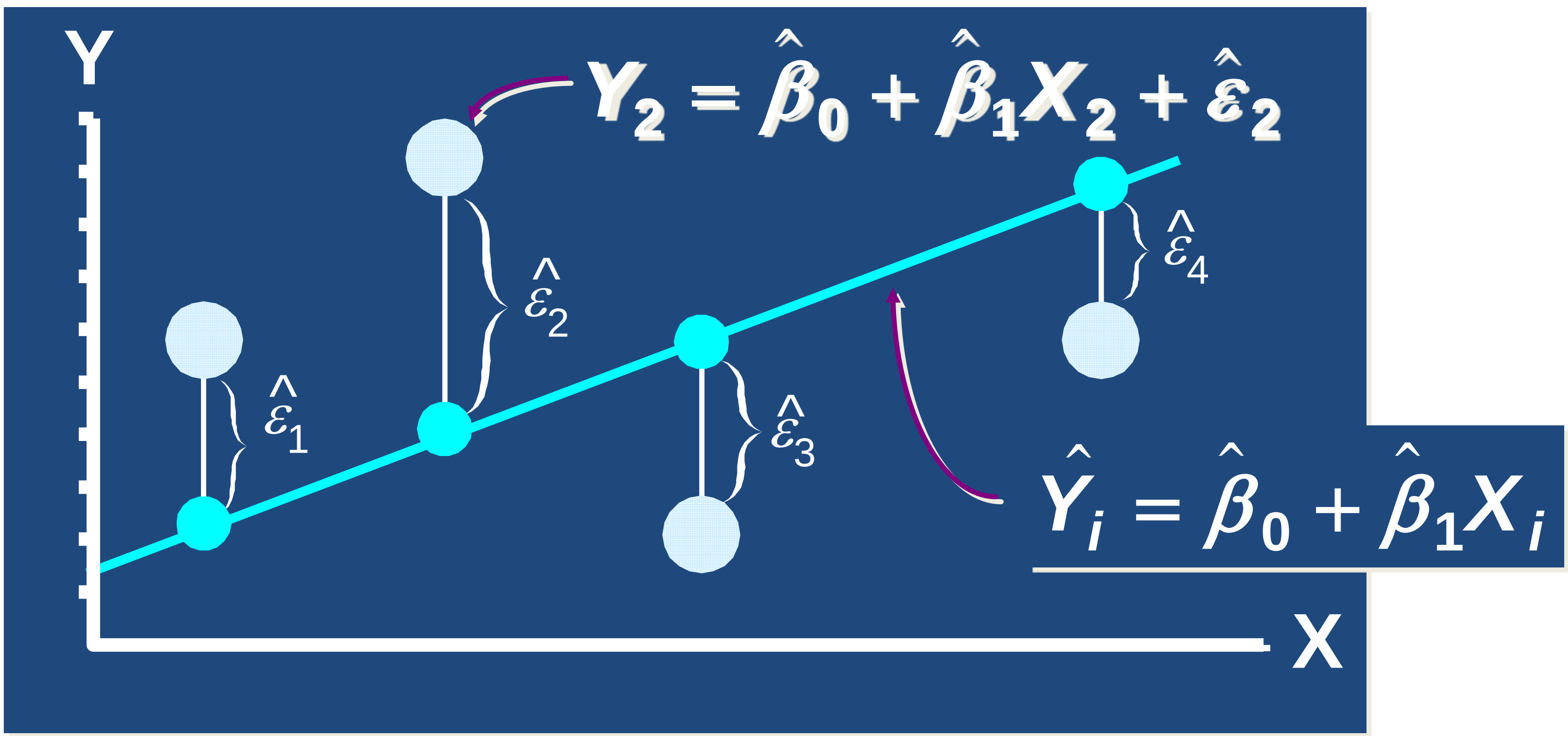
$$\sum_{i=1}^{n}(Y_i - b_0 - b_1 X_i)^2$$

Take derivative

Set to 0 and solve

Least Squares (LS) Line Graphically

LS minimizes $\sum_{i=1}^{n} \hat{\varepsilon}_i^2 = \hat{\varepsilon}_1^2 + \hat{\varepsilon}_2^2 + \hat{\varepsilon}_3^2 + \hat{\varepsilon}_4^2$



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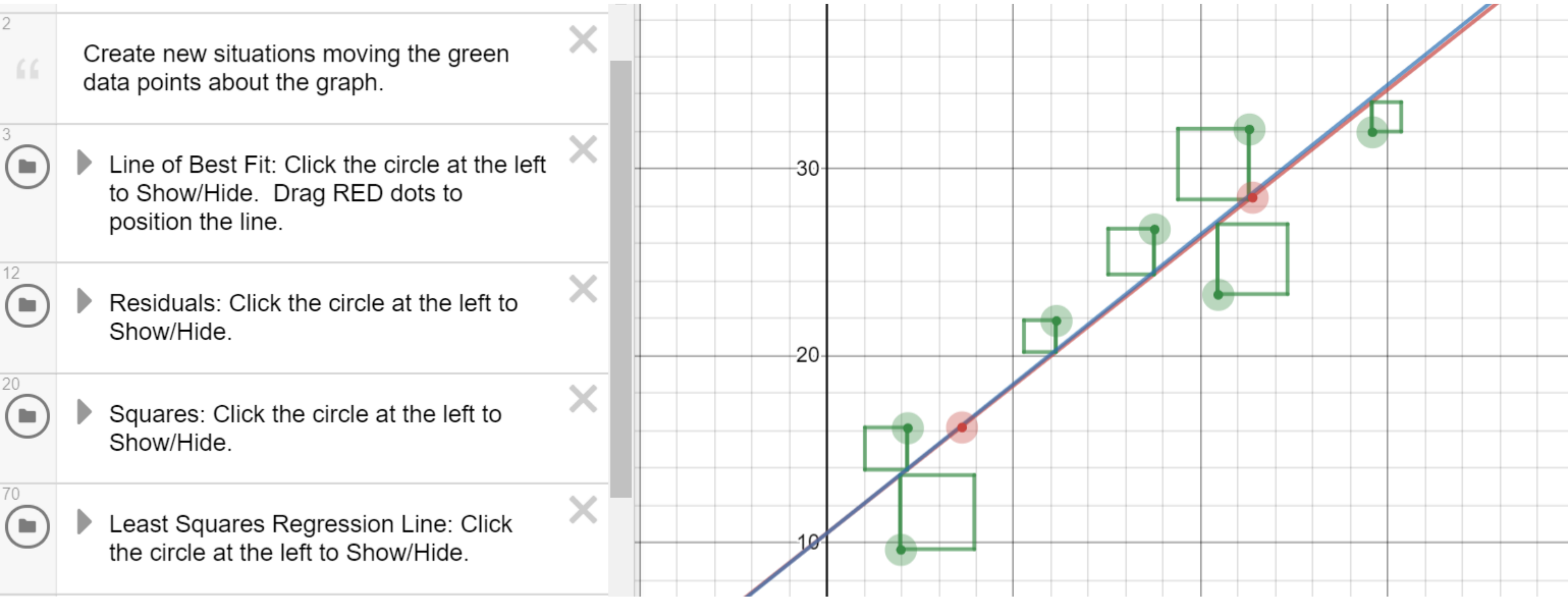
Least Squares Line Graphically – Interactive Demo



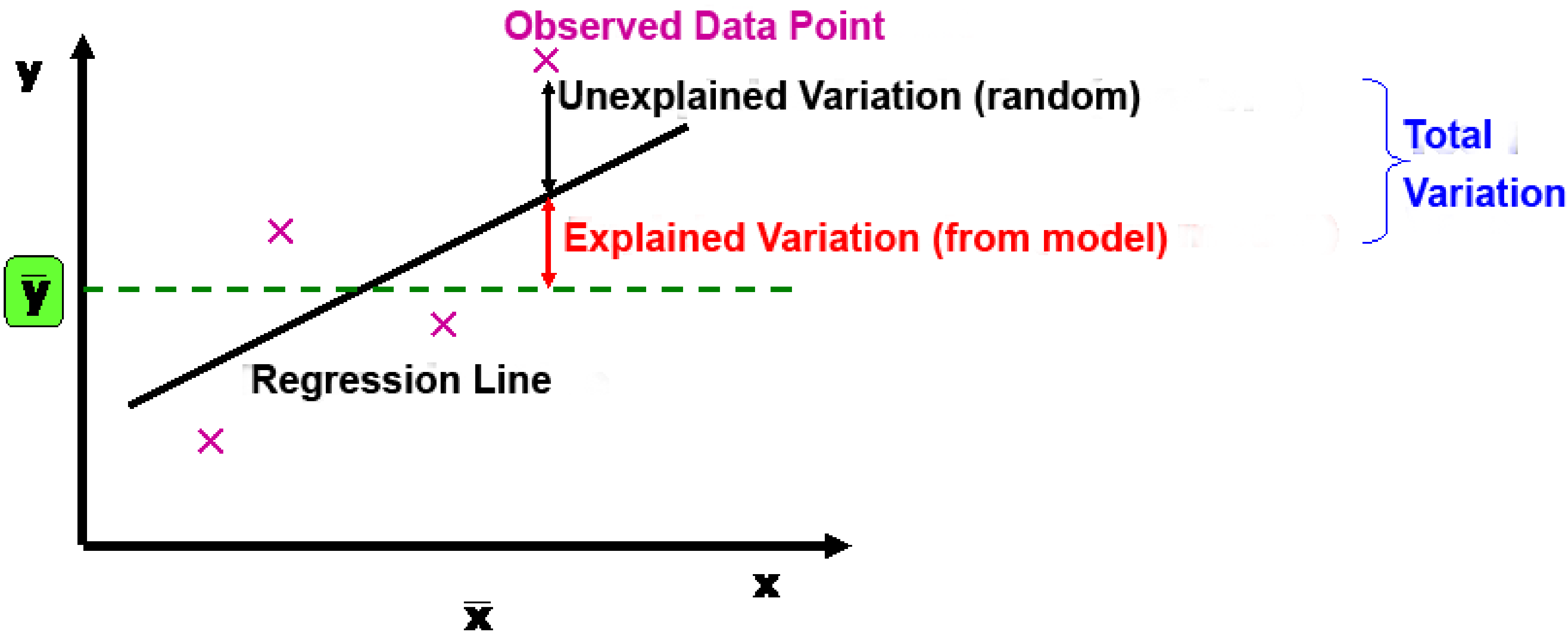
https://www.desmos.com/calculator/zvrc4lg3cr

Try it out yourself!
https://web.cs.wpi.edu/~imgd2905/d24/groupwork/12-least-squares/handout.html

Outline

- Introduction (done)
- Simple Linear Regression (done)
- Measures of Variation (next)
  - Coefficient of Determination
  - Correlation
- Misc

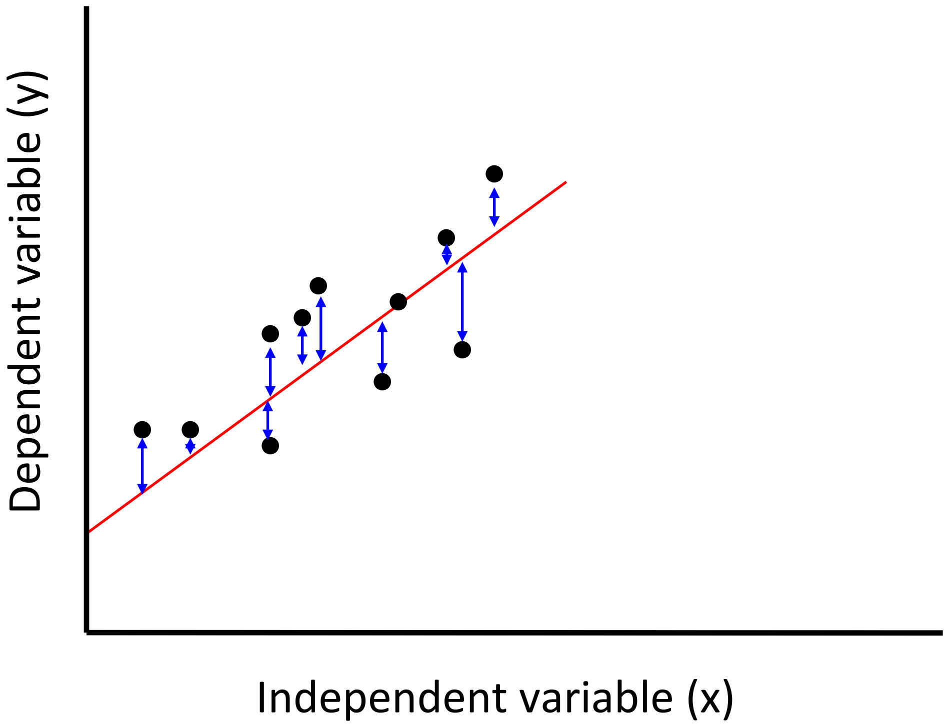
Measures of Variation



- Several sources of variation in y
  - Error in prediction (unexplained)
  - Variation from model (explained)

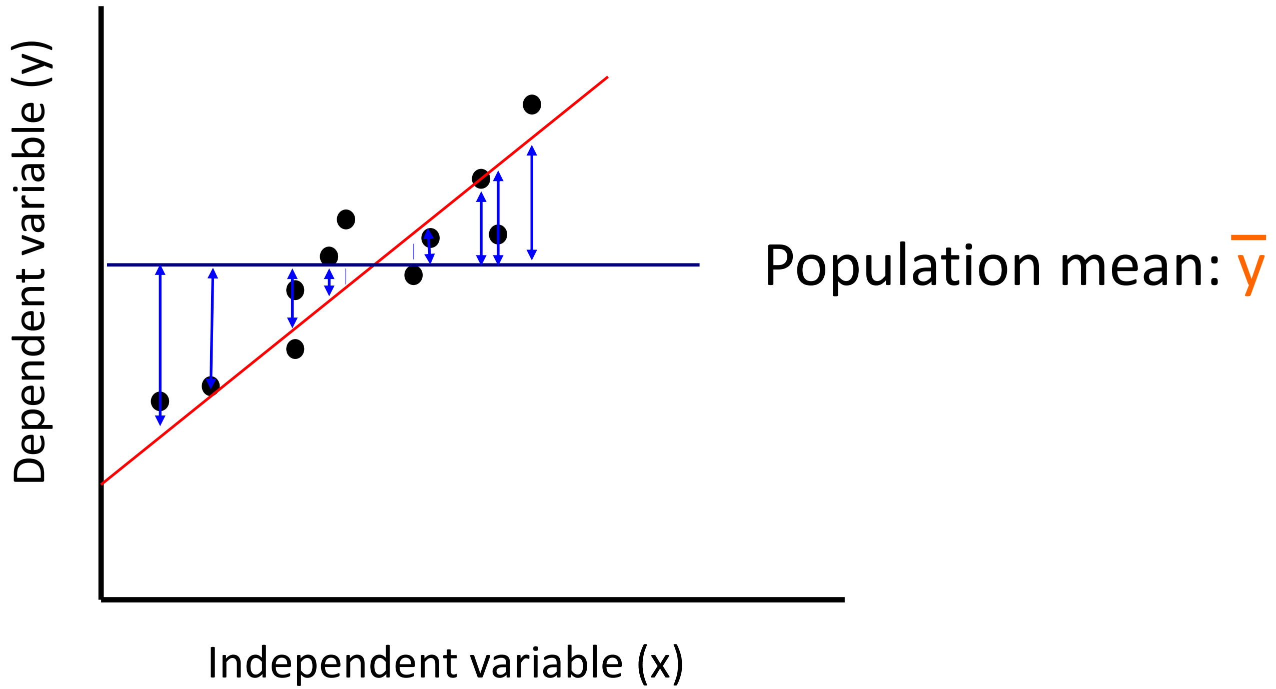
Break this down (next)

Sum of Squares of Error (SSE)



- Least squares regression selects line with lowest total sum of squared prediction errors
- Sum of Squares of Error, or SSE
- Measure of unexplained variation

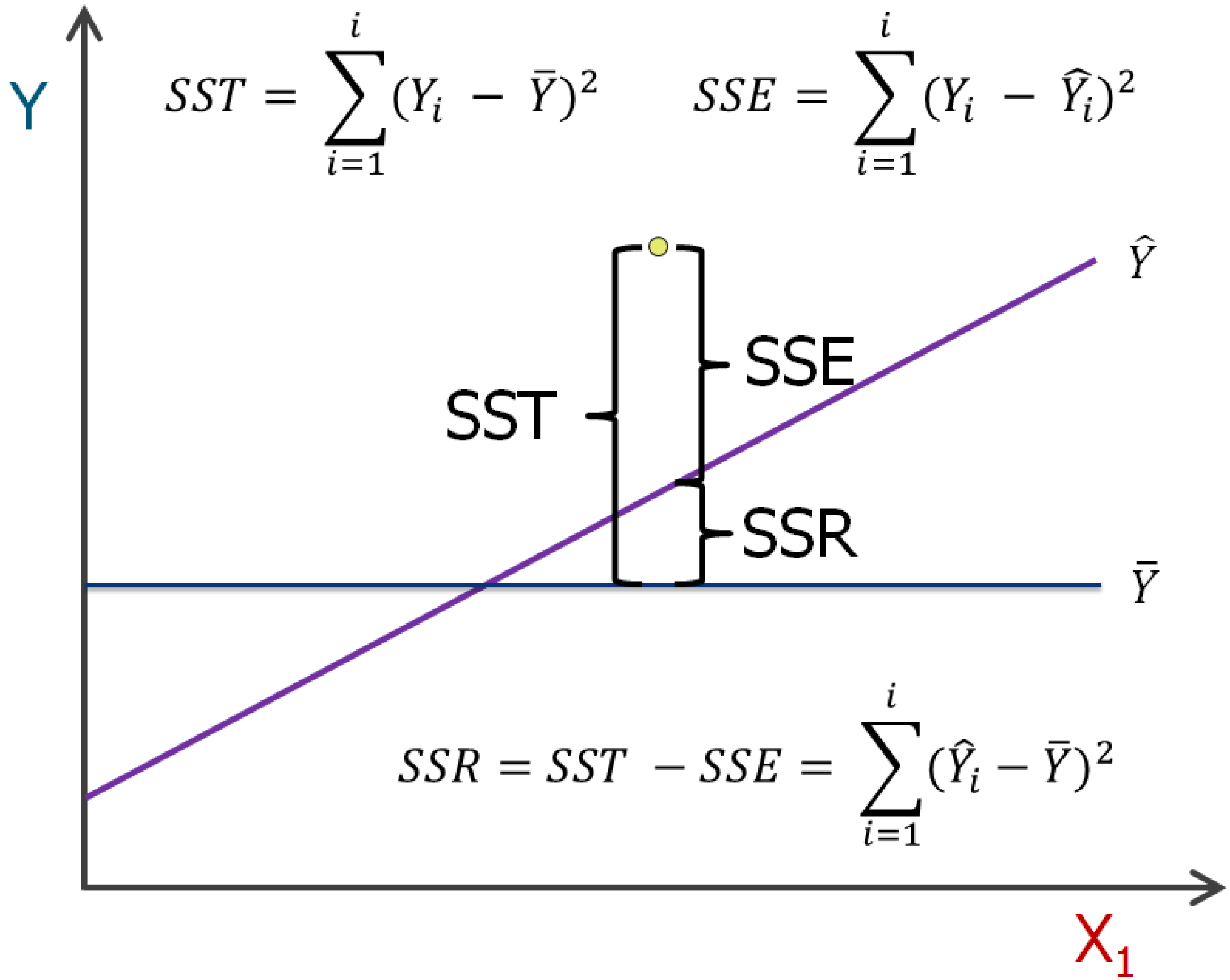
Sum of Squares Regression (SSR)



- Differences between prediction and population mean
  - Gets at variation due to X & Y
- Sum of Squares Regression, or SSR
- Measure of explained variation

Sum of Squares Total

- Total Sum of Squares, or SST = SSR + SSE



Coefficient of Determination

- Proportion of total variation (SST) explained by the regression (SSR) is known as the Coefficient of Determination (R^2)

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

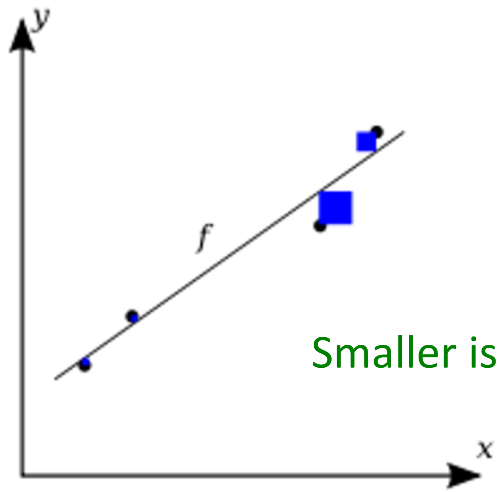
- Ranges from 0 to 1 (often said as a percent)

  1 – regression explains all of variation

  0 – regression explains none of variation

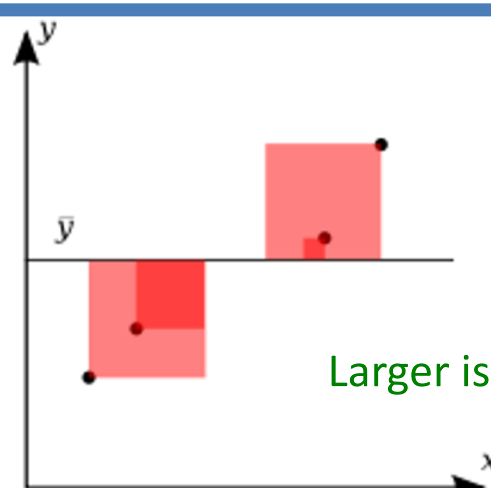
Coefficient of Determination – Visual Representation

$R^2 = 1 -$



Smaller is better

Variation in observed data model cannot explain (error)



Larger is better

Total variation in observed data

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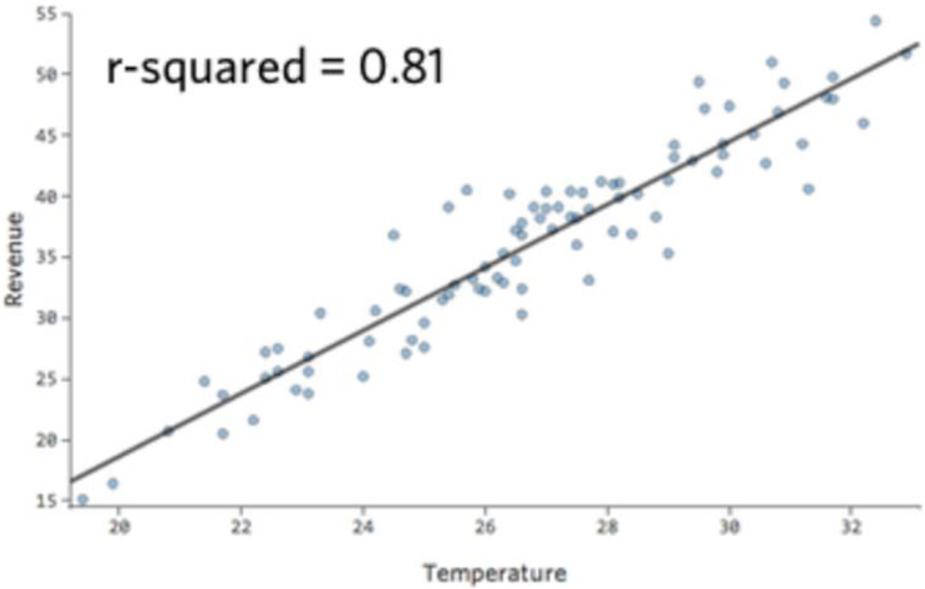
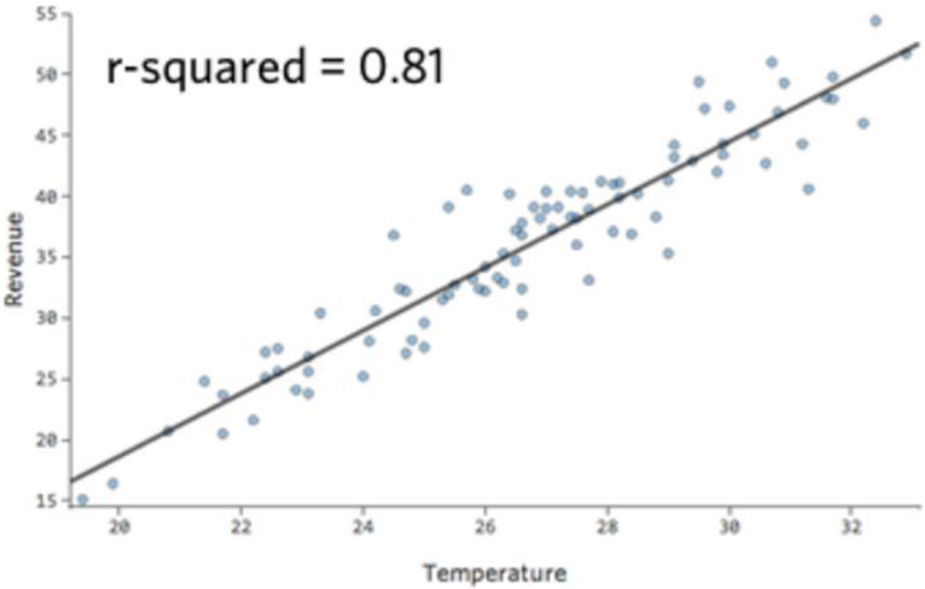
Coefficient of Determination Example

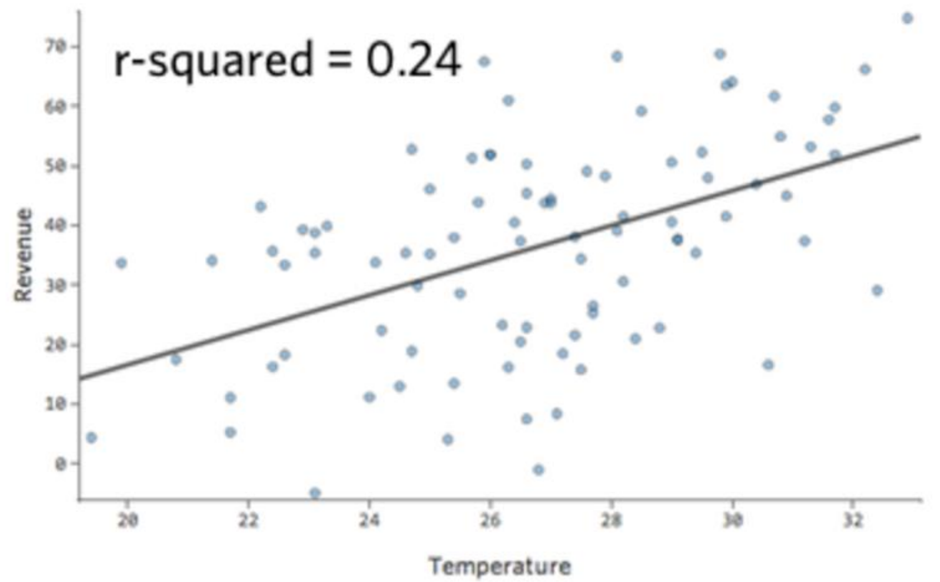


- How "good" is regression model? Roughly:

  $0.8 \leq R^2 \leq 1$ strong

Coefficient of Determination Example



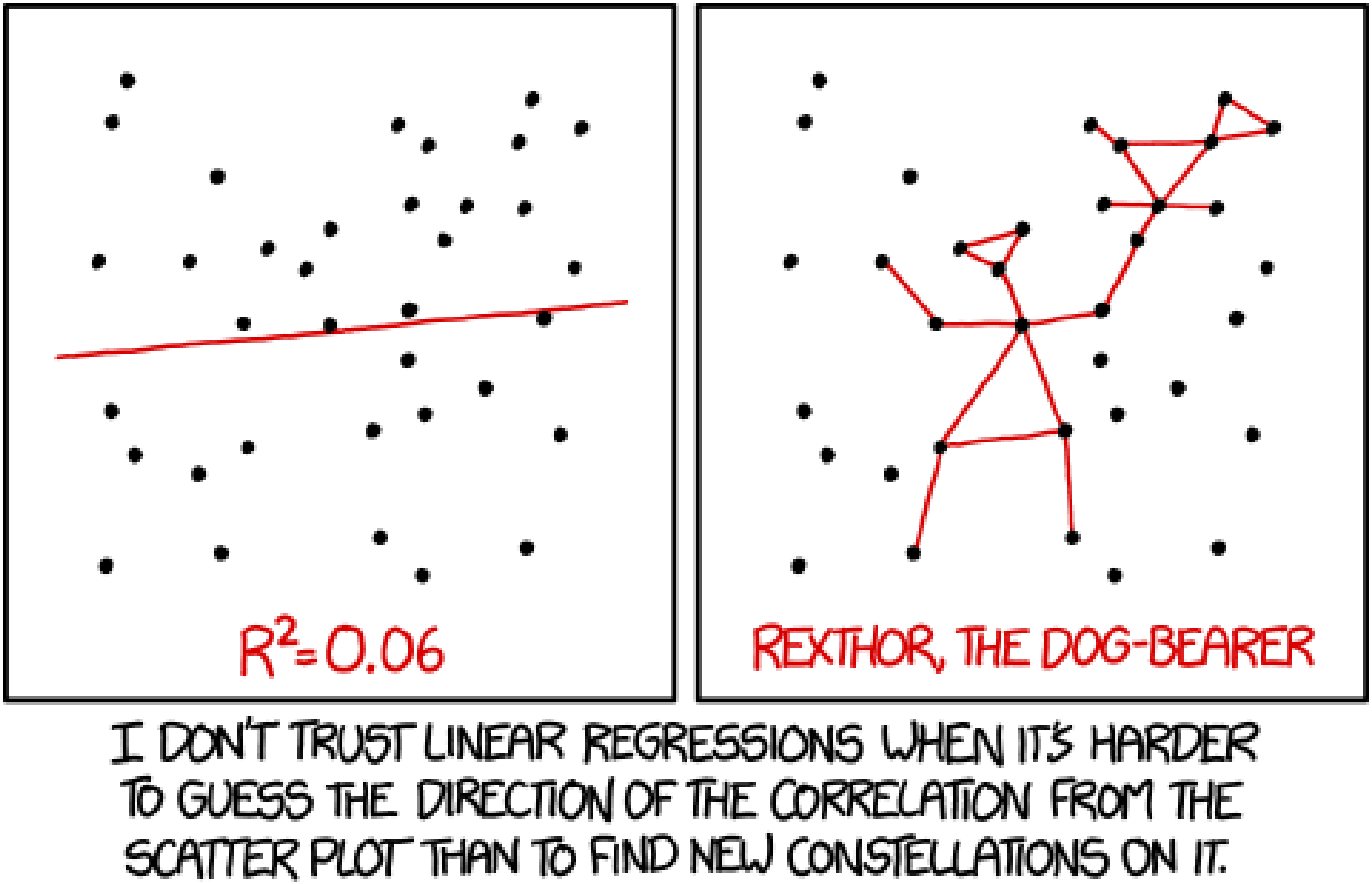


- How "good" is regression model? Roughly:

$0.8 \le R^2 \le 1$ strong

$0 \le R^2 < 0.5$ weak

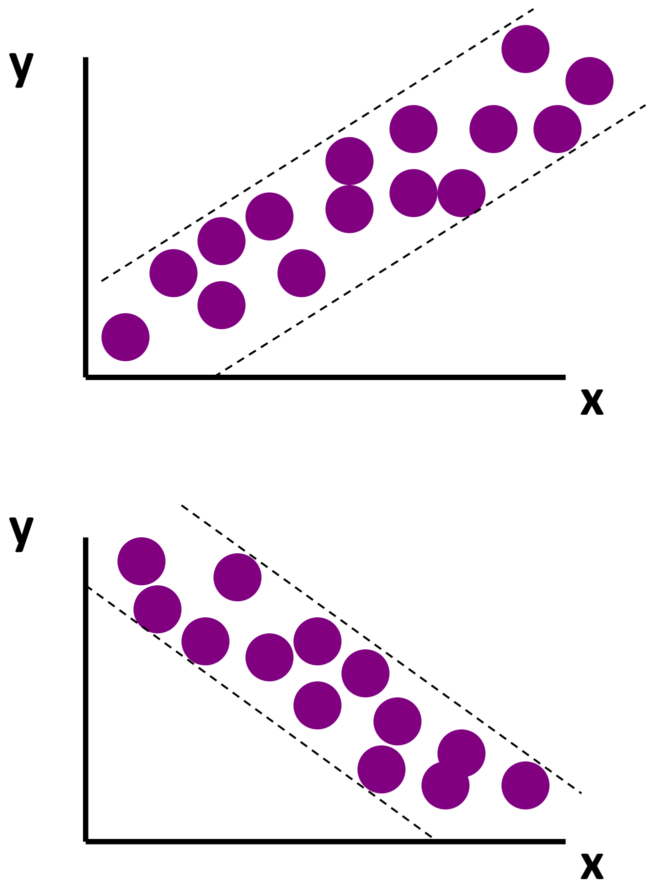
How “Good” is the Regression Model?



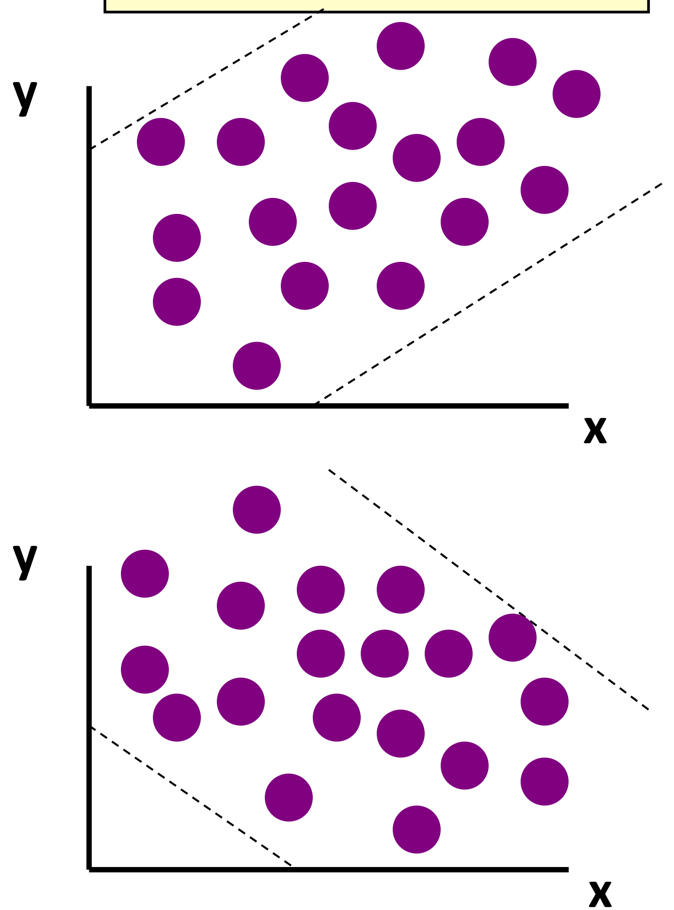
https://xkcd.com/1725/

Relationships Between X & Y

Strong relationships



Weak relationships



Relationship Strength and Direction – Correlation

- Correlation measures strength and direction of linear relationship
  - -1 perfect neg. to +1 perfect pos.
  - – Sign is same as regression slope
  - – Denoted R. Why? Square R = R^2

Pearson's Correlation Coefficient

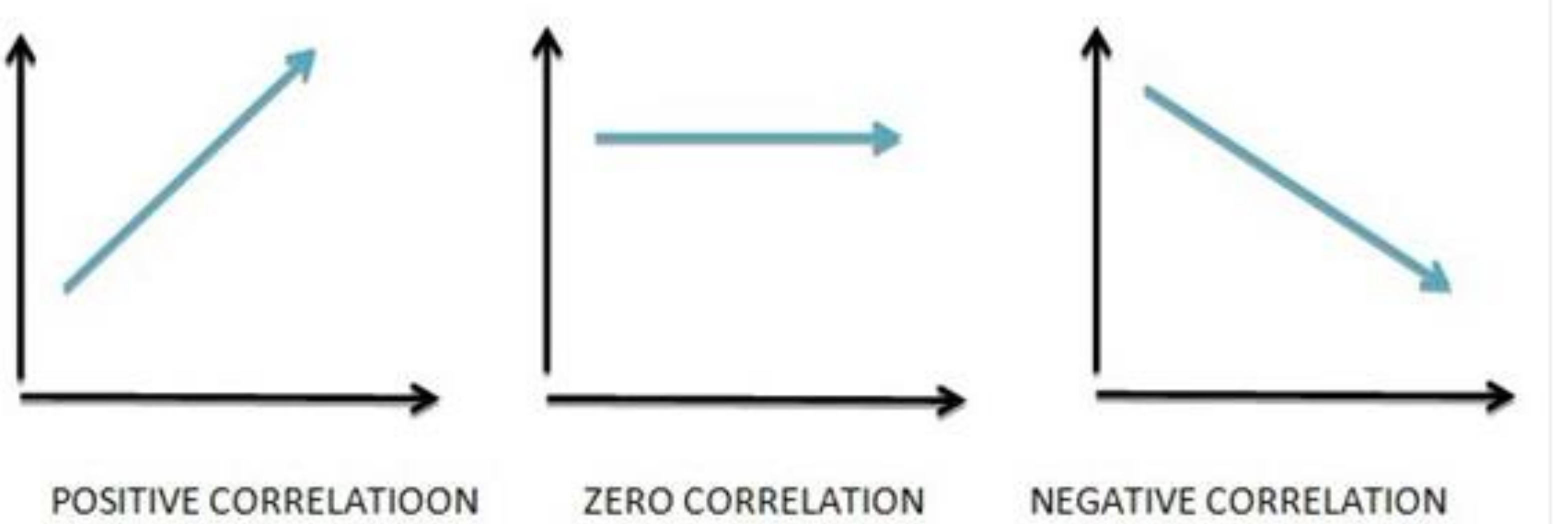
$$r = \frac{\sum(X-\overline{X})(Y-\overline{Y})}{\sqrt{\sum(X-\overline{X})^2}\sqrt{(Y-\overline{Y})^2}}$$

Vary together

Vary Separately

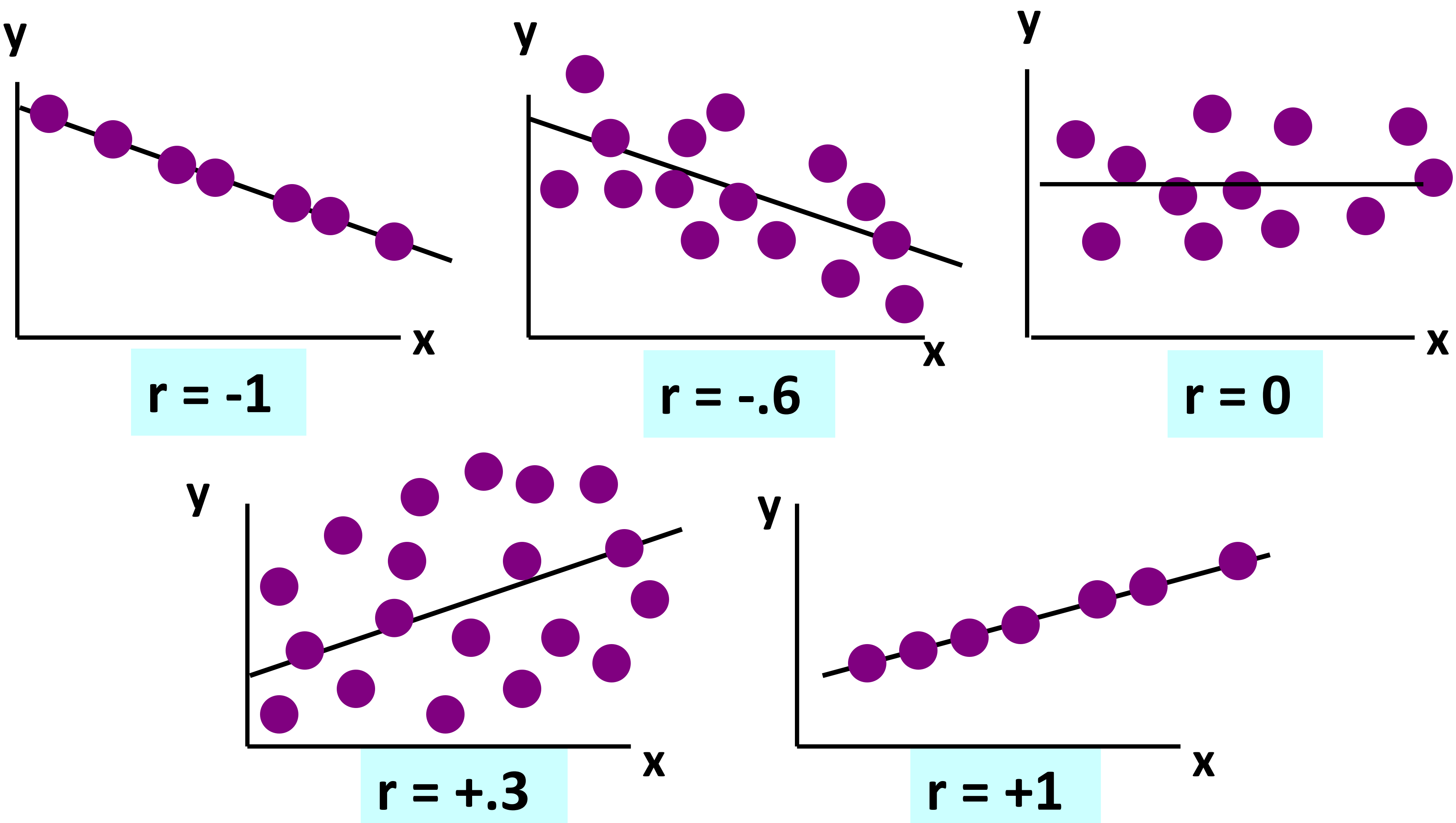
Where, $\overline{X}$ = mean of X variable

$\overline{Y}$ = mean of Y variable



https://www.mbaskool.com/2013_images/stories/dec_images/pearson-coeff-bcon.jpg

Correlation Examples

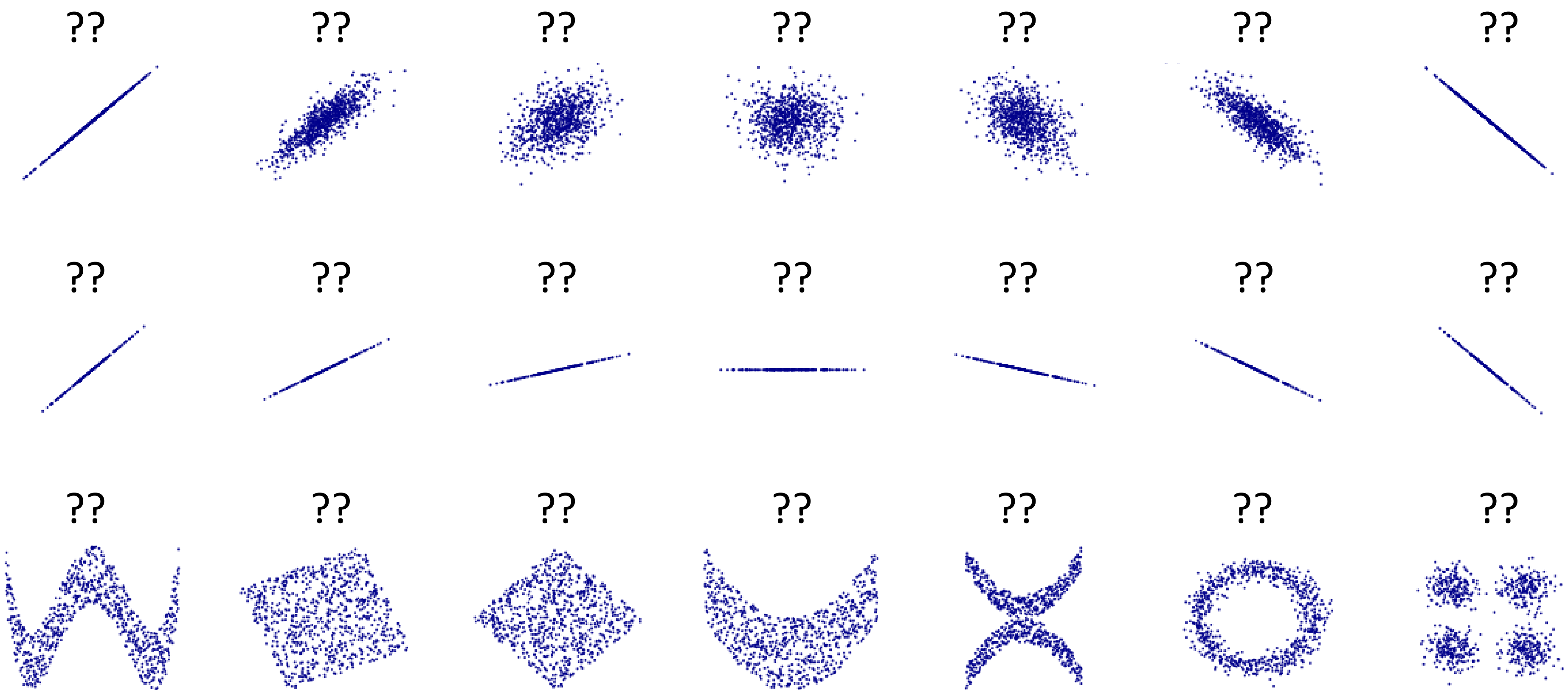


Groupwork

- Introduction
  - **Icebreaker**: What game are you looking forward to playing this summer?
- Groupwork
  - Think, discuss, write down – qualtrics
- Correlation
  - Consider scatterplots
  - Estimate correlation

https://web.cs.wpi.edu/~ht tps://web.cs.wpi.edu/~img d2905/d24/groupwork/13-correlation/handout.html/handout.html

Correlation Examples



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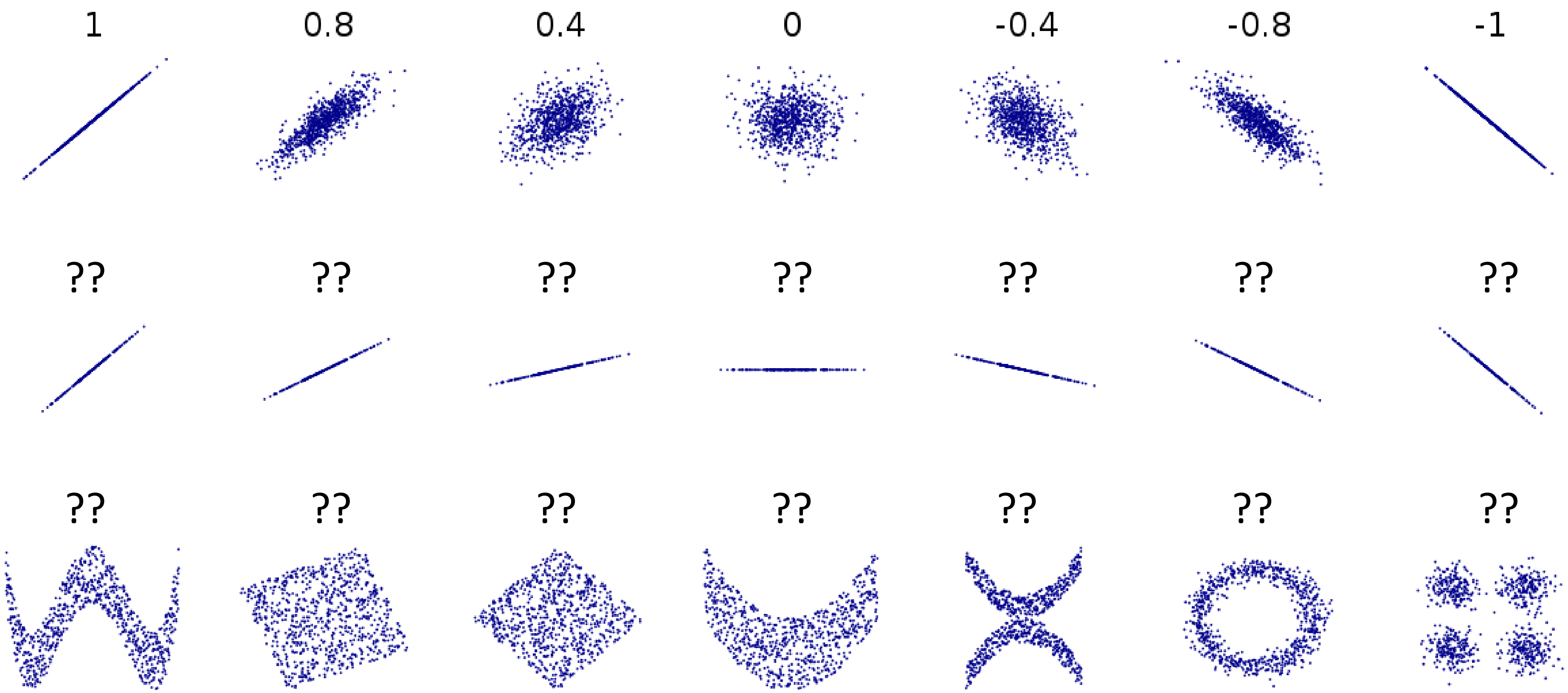
Correlation Examples



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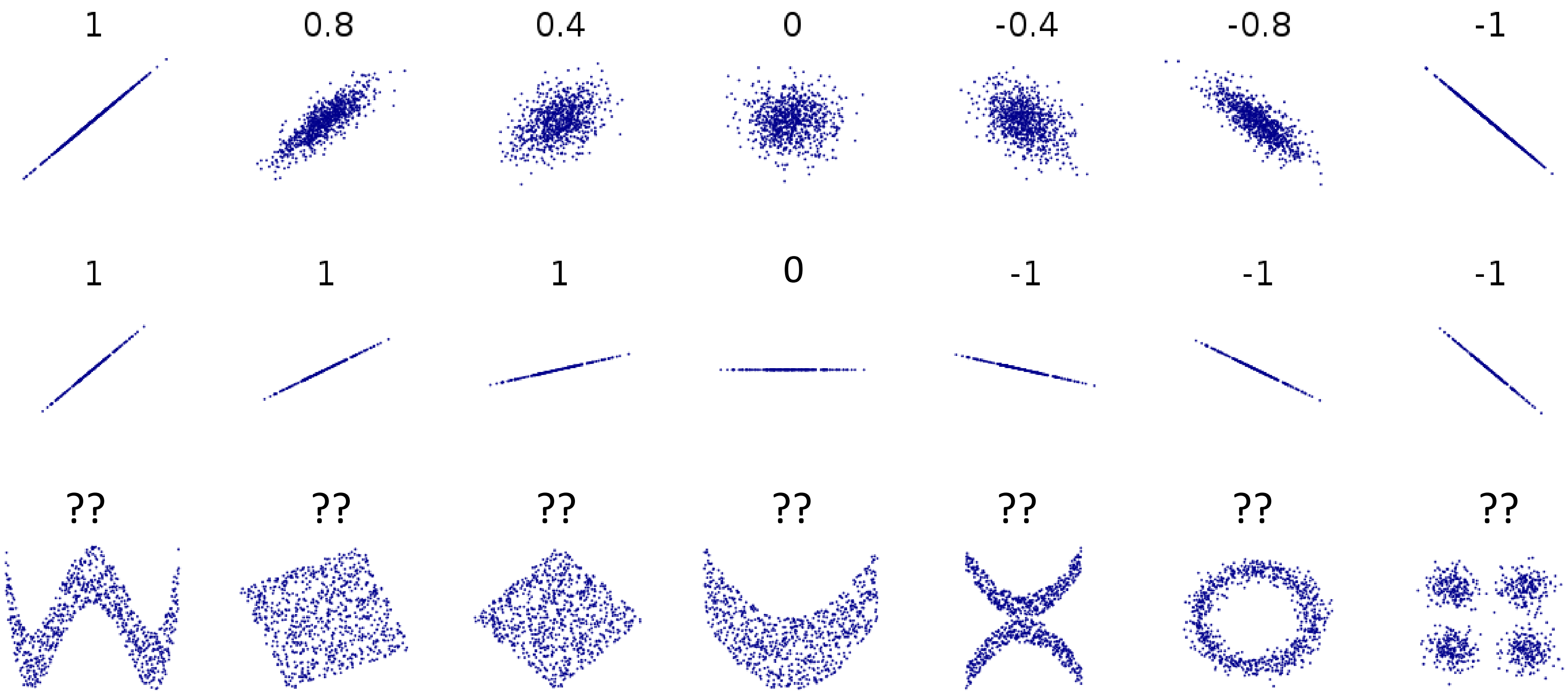
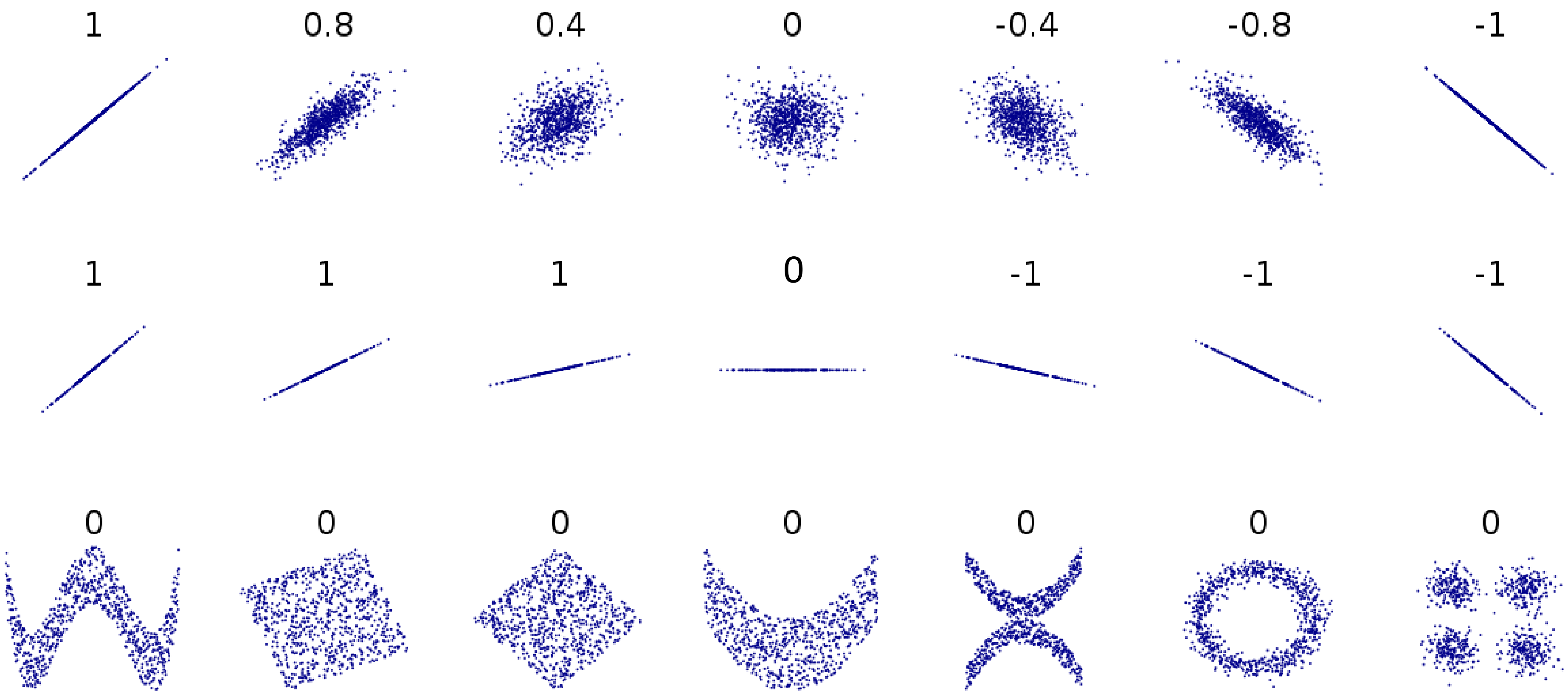


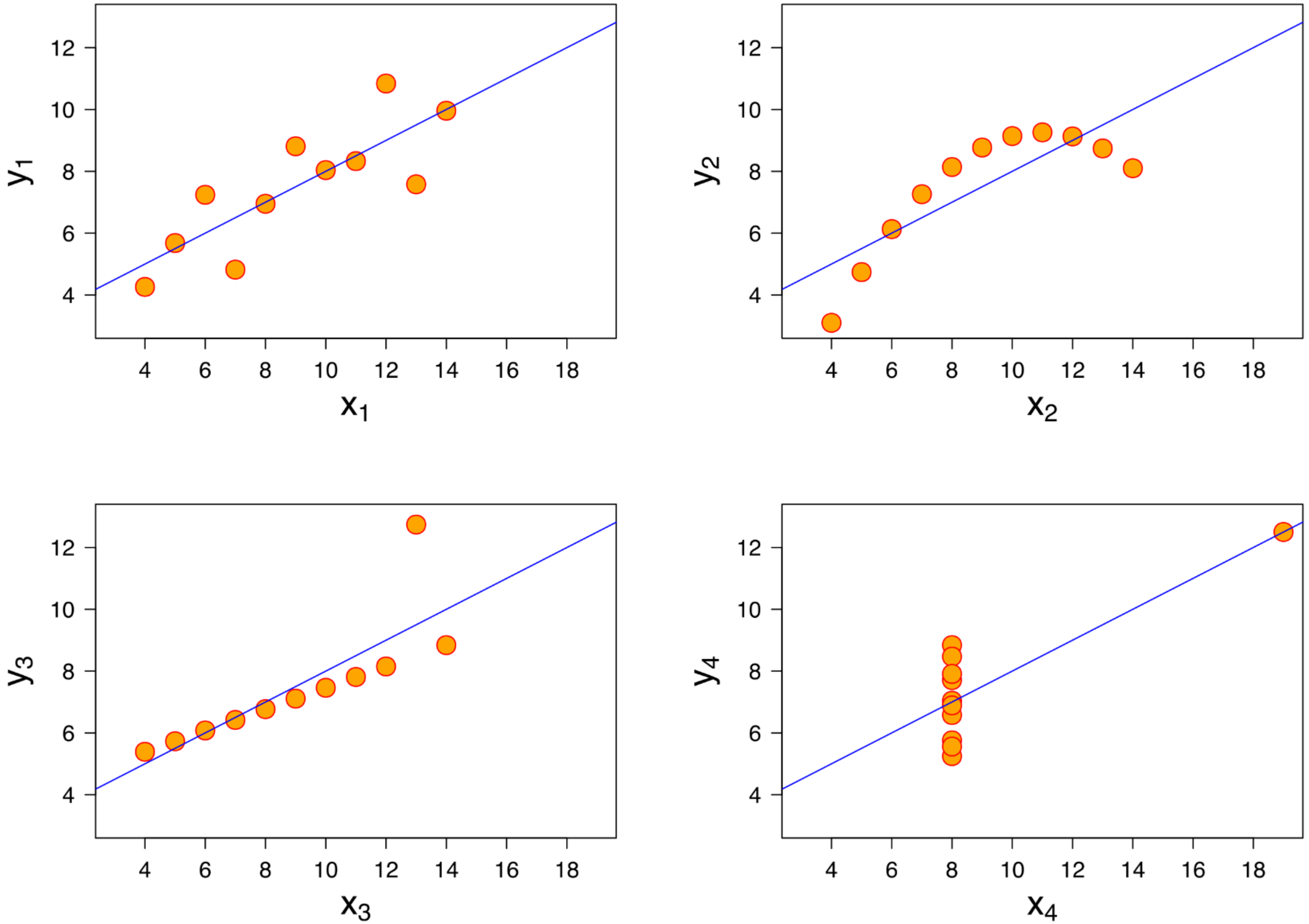
https://upload.wikimedia.org/wikipedia/commons/thumb/d/d4/Correlation_examples2.svg/1200px-Correlation_examples2.svg.png

Correlation Examples



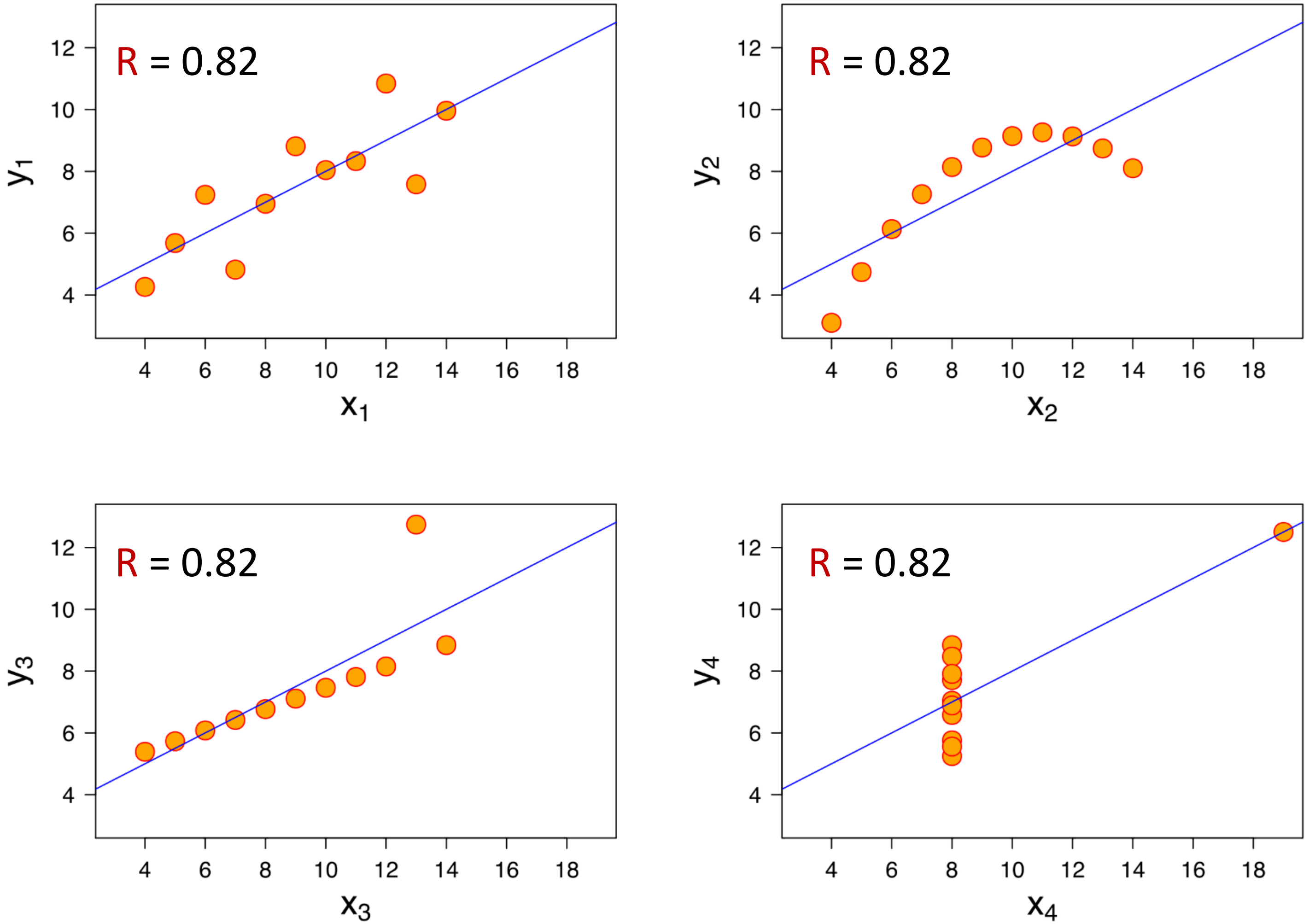
https://upload.wikimedia.org/wikipedia/commons/thumb/d/d4/Correlation_examples2.svg/1200px-Correlation_examples2.svg.png

Correlation Examples



(Note, would want to use residual analysis before using predictions!)

Correlation Examples



(Note, would want to use residual analysis before using predictions!)

Anscombe's Quartet

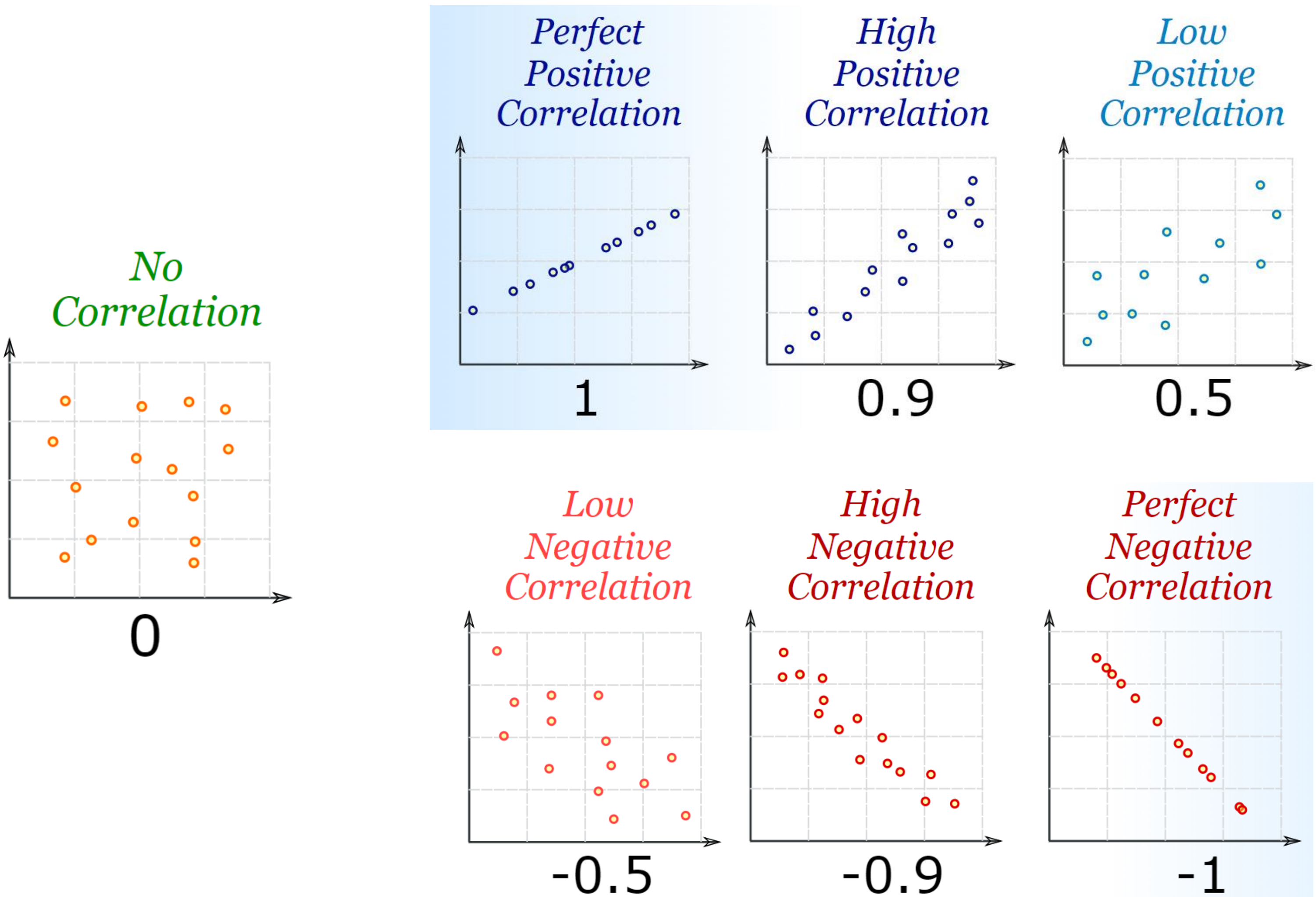
https://en.wikipedia.org/wiki/Anscombe%27s_quartet

Summary stats:

$Mean_x$	9
$Mean_y$	7.5
Var_x	11
Var_y	4.125
Model:	y=0.5x+3

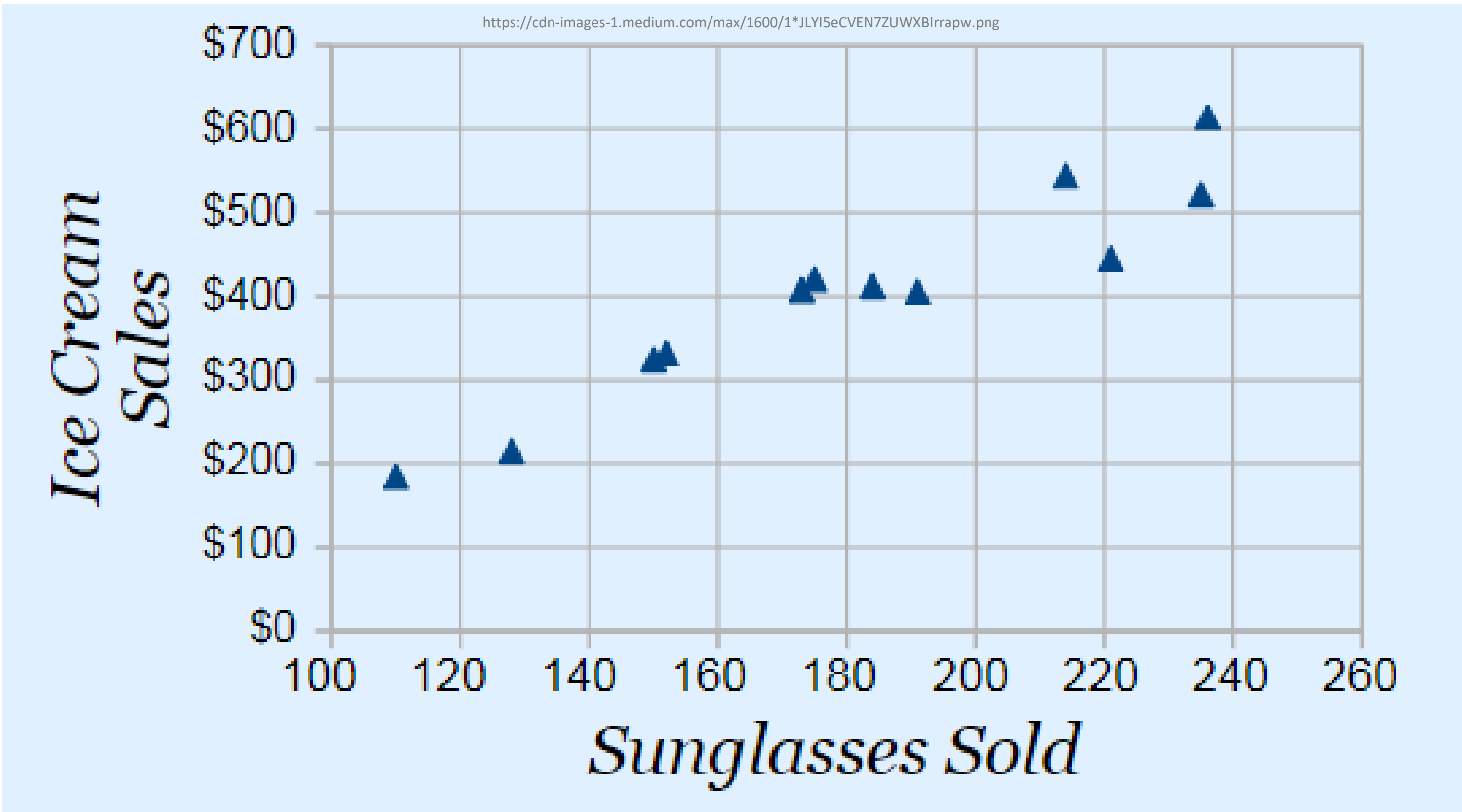
$R^2 = 0.69$

Correlation Summary



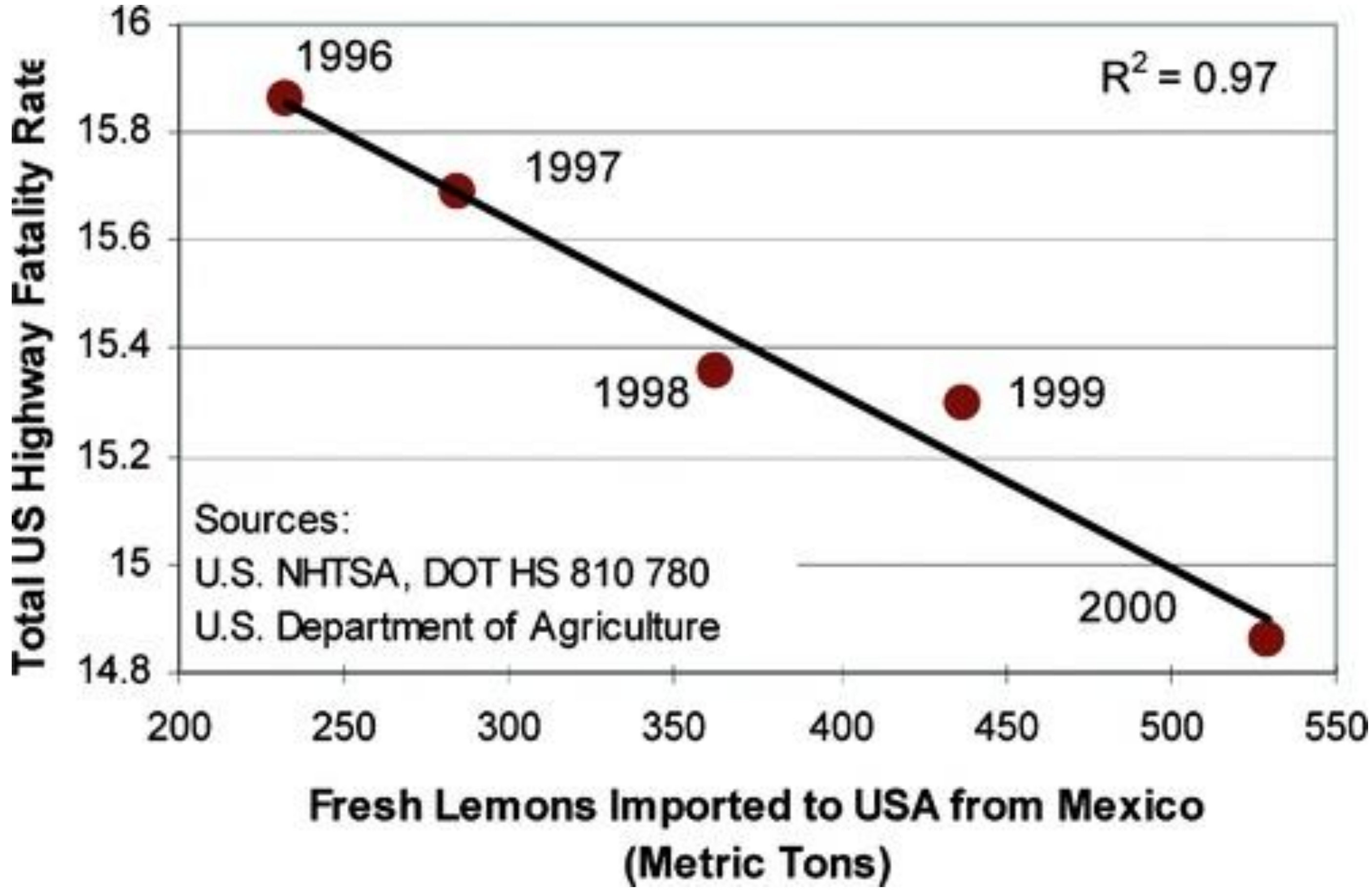
https://www.mathsisfun.com/data/correlation.html

Correlation is not Causation



Buying sunglasses *causes* people to buy ice cream?

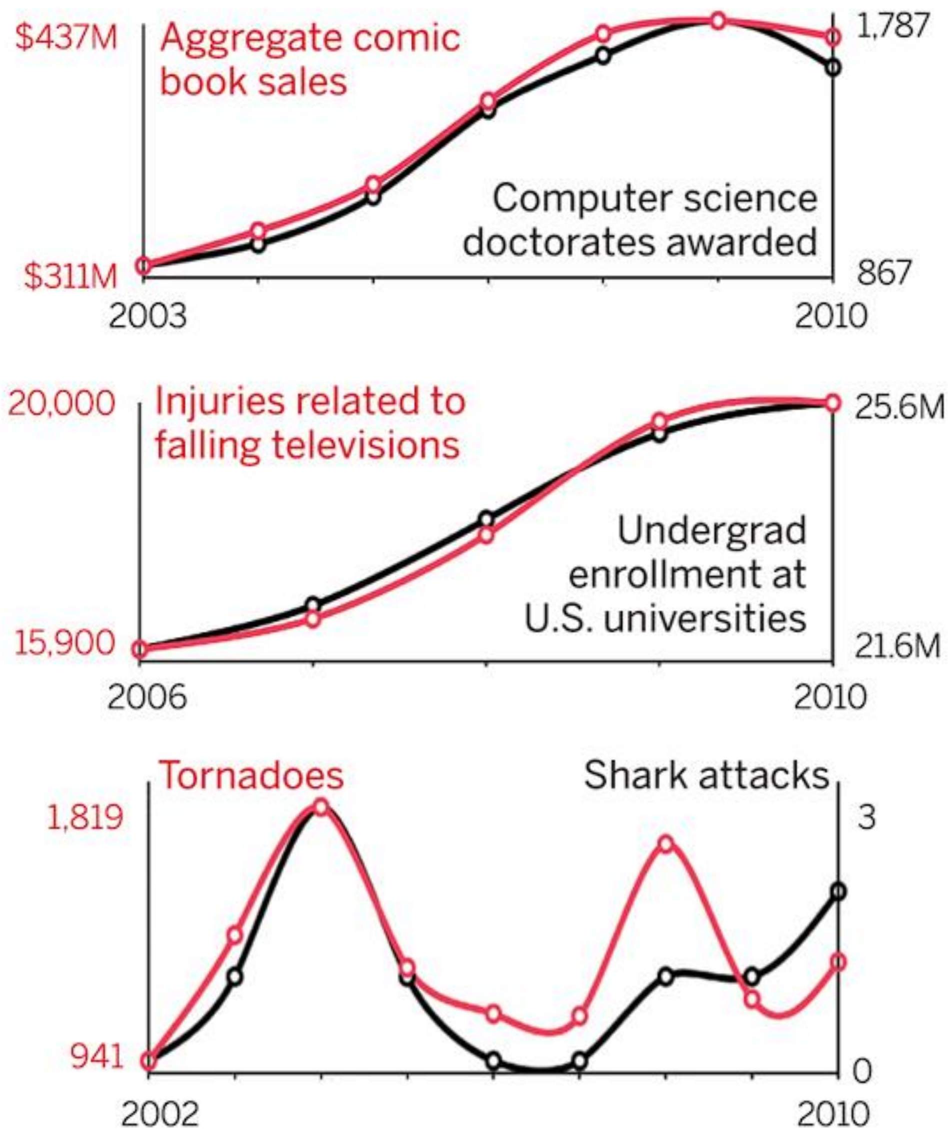
Correlation is not Causation



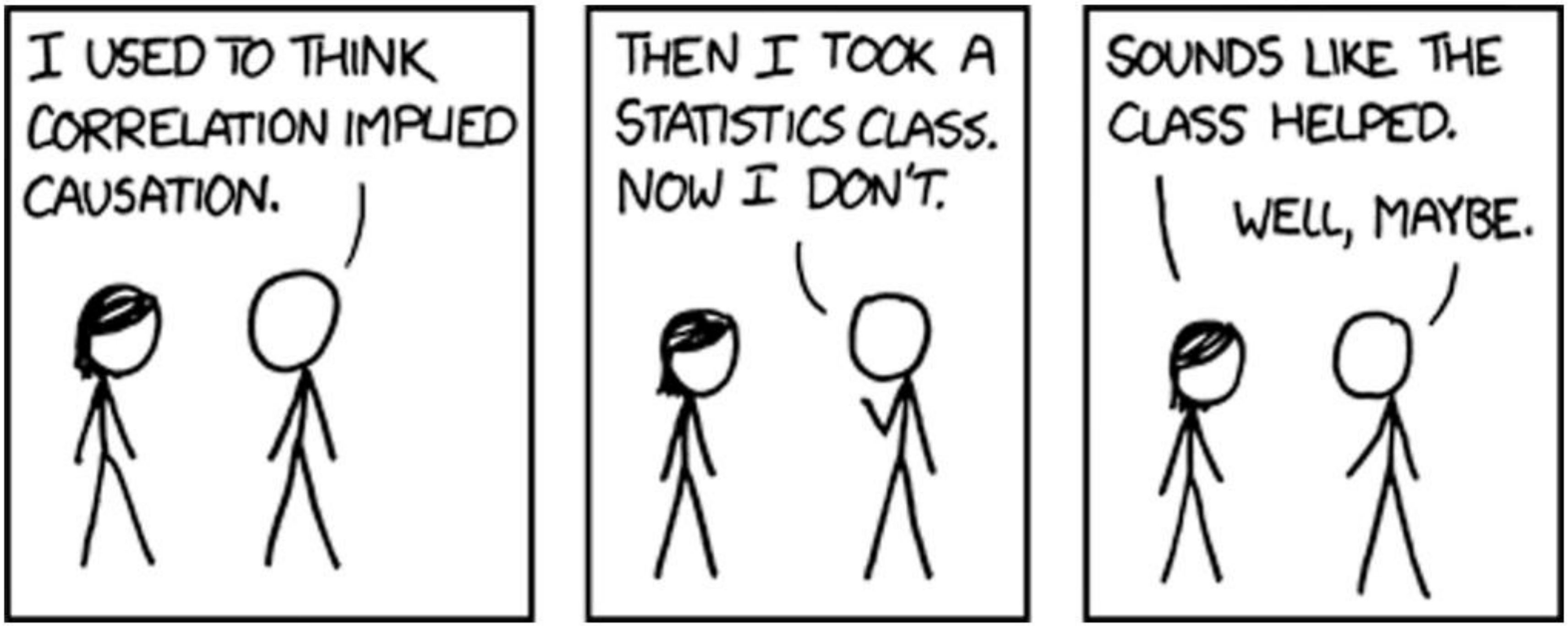
Importing lemons causes fewer highway fatalities?

Correlation is not Causation

https://science.sciencemag.org/content/sci/348/6238/980.2/F1.large.jpg?width=800&height=600&carousel=1



Correlation is not Causation



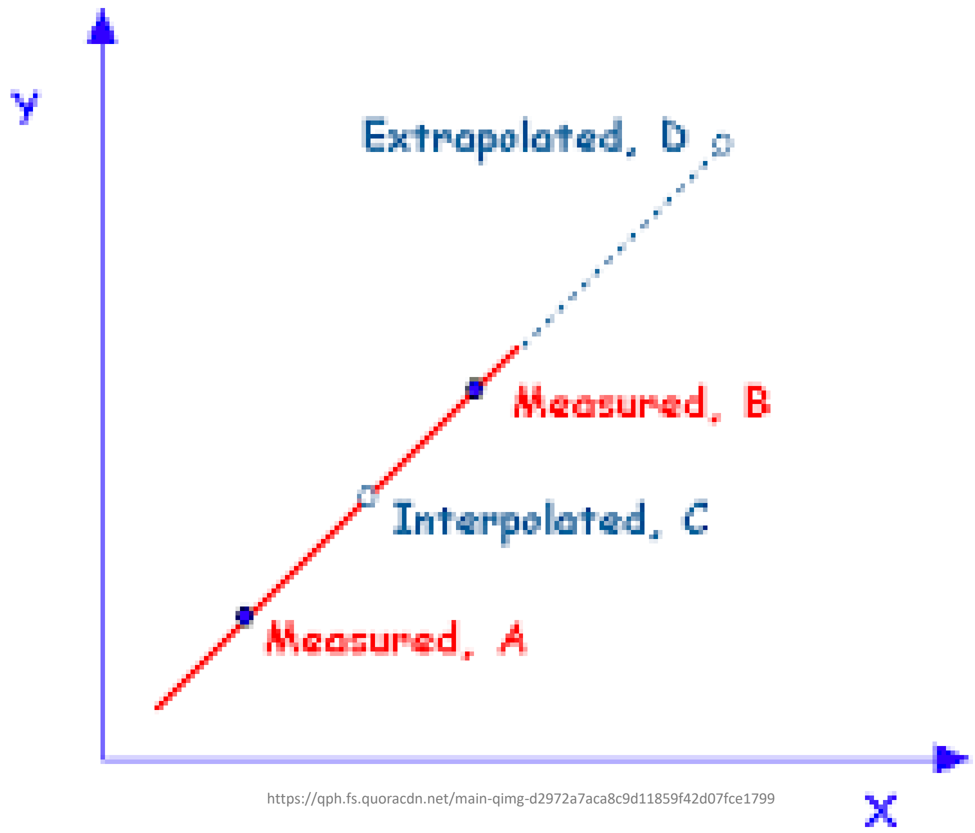
https://xkcd.com/552/

Outline

- Introduction (done)
- Simple Linear Regression (done)
- Measures of Variation (done)
- Misc (next)

Extrapolation versus Interpolation

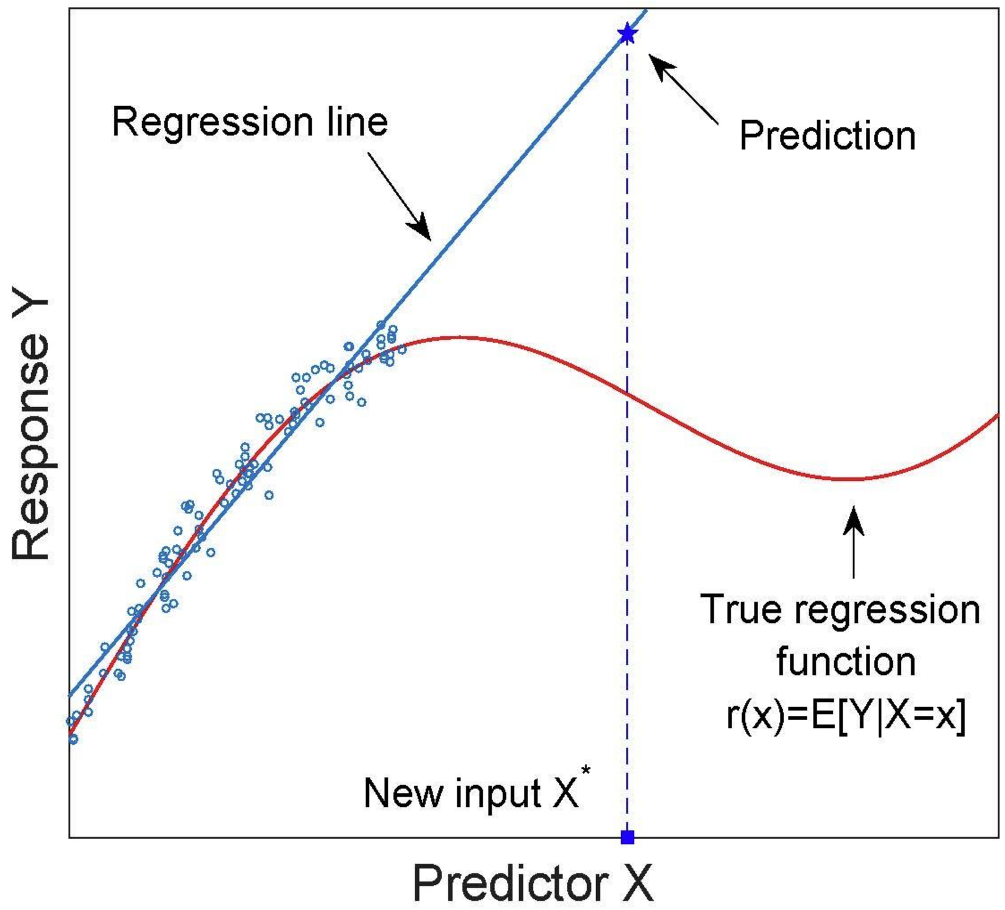
- Prediction
  - Interpolation – within measured X-range
  - Extrapolation – outside measured X-range



https://qph.fs.quoracdn.net/main-qimg-d2972a7aca8c9d11859f42d07fce1799

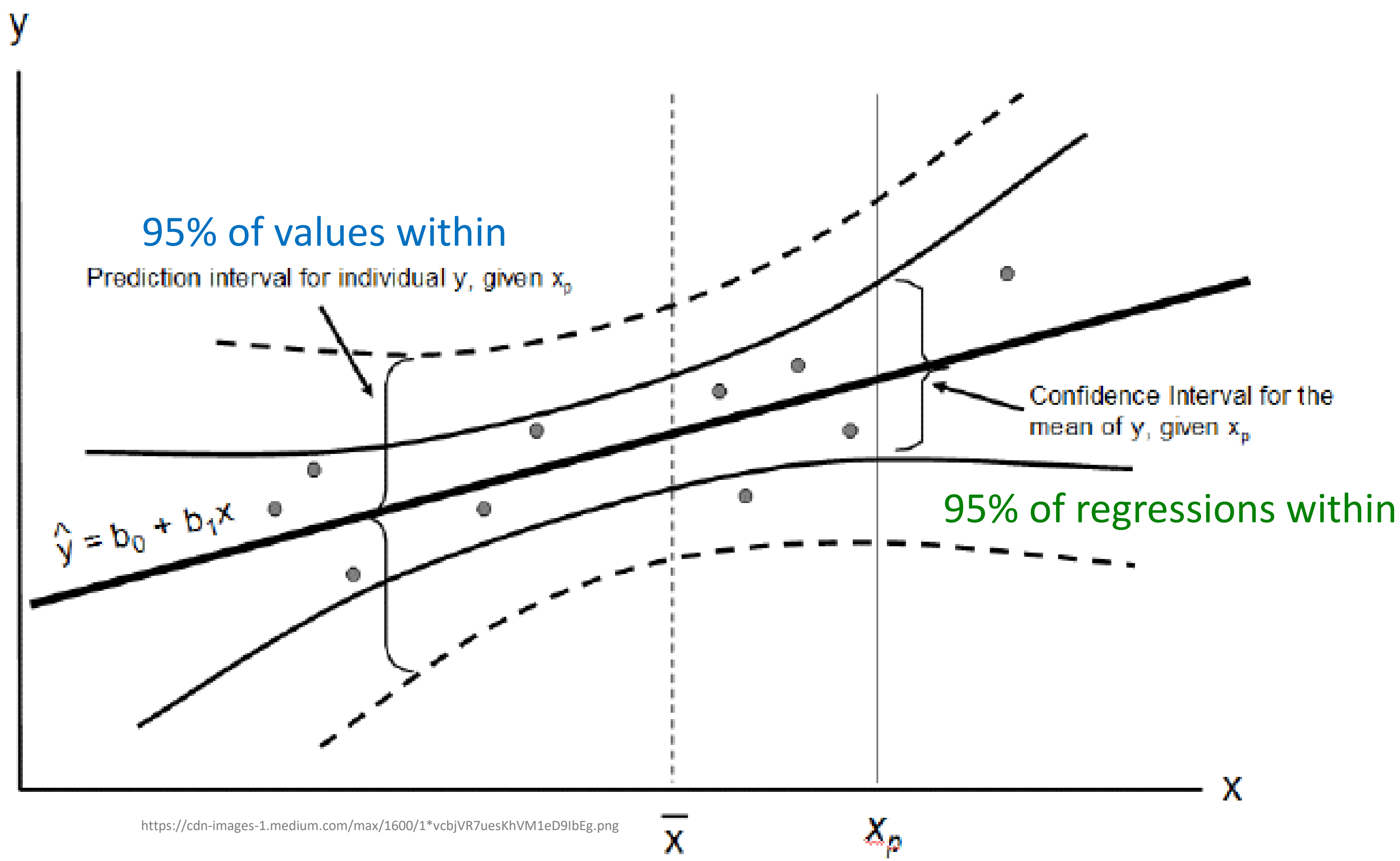
Be Careful When Extrapolating

https://i.stack.imgur.com/3Ab7e.jpg



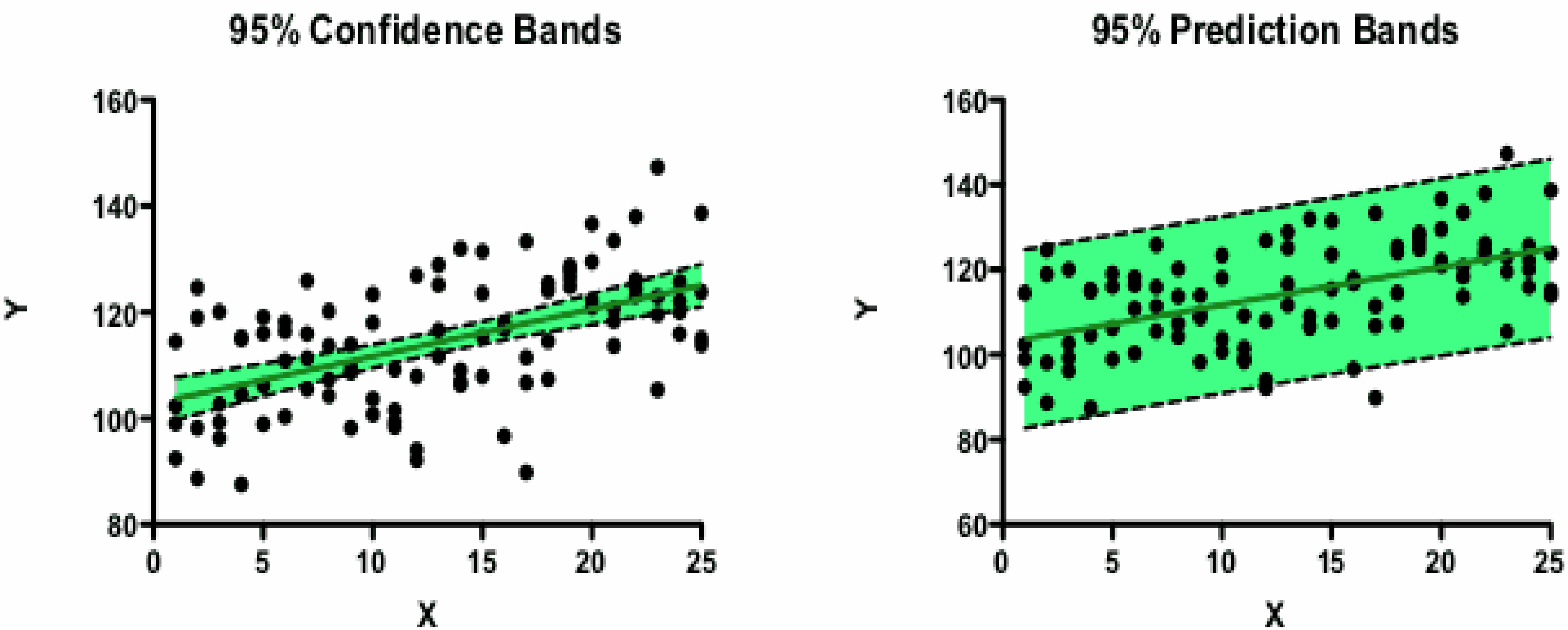
If extrapolate, make sure have reason to assume model continues

Prediction and Confidence Intervals (1 of 2)



https://cdn-images-1.medium.com/max/1600/1*vcbjVR7uesKhVM1eD9IbEg.png

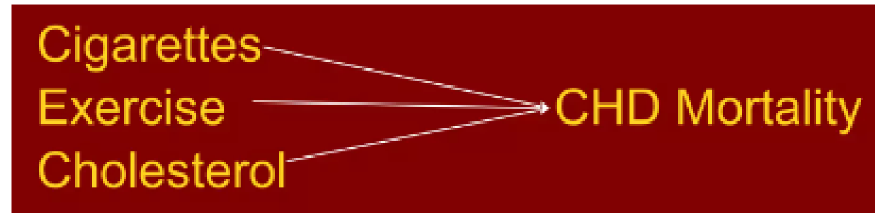
Prediction and Confidence Intervals (2 of 2)



https://www.graphpad.com/guides/prism/7/curve-fitting/reg_mostpointsareoutsideconfidencebands.png

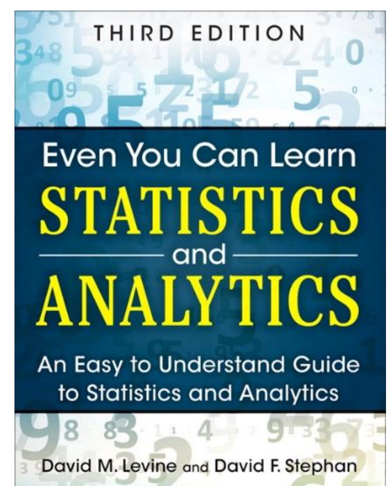
Multiple Independent Variables

- Chronic heart disease (CHD) correlates with smoking
  - $R^2 = 0.5$
- But what about other 50%
- Correlation with exercise? Cholesterol?



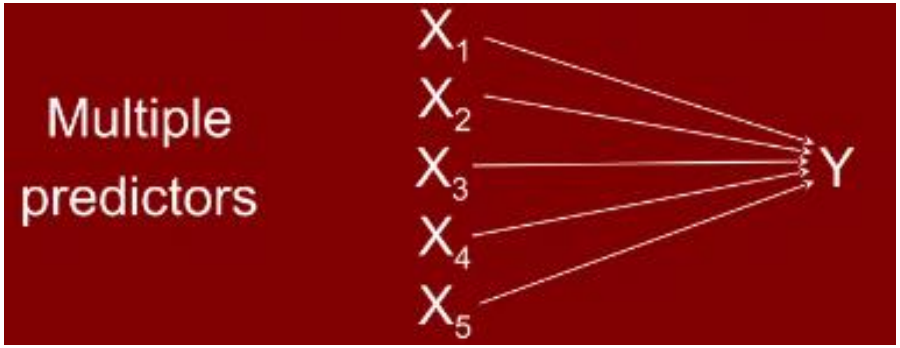
Multiple Linear Regression

Chapter 11



Single Linear Regression → Multiple Linear Regression

- Use several independent variables to predict dependent variable
- Weights each predictor based on strength of relationship
- Makes adjustments for inter-relationships among predictors
- Gives overall fit (R^2)
- Note: Need independent variables not highly related to each other

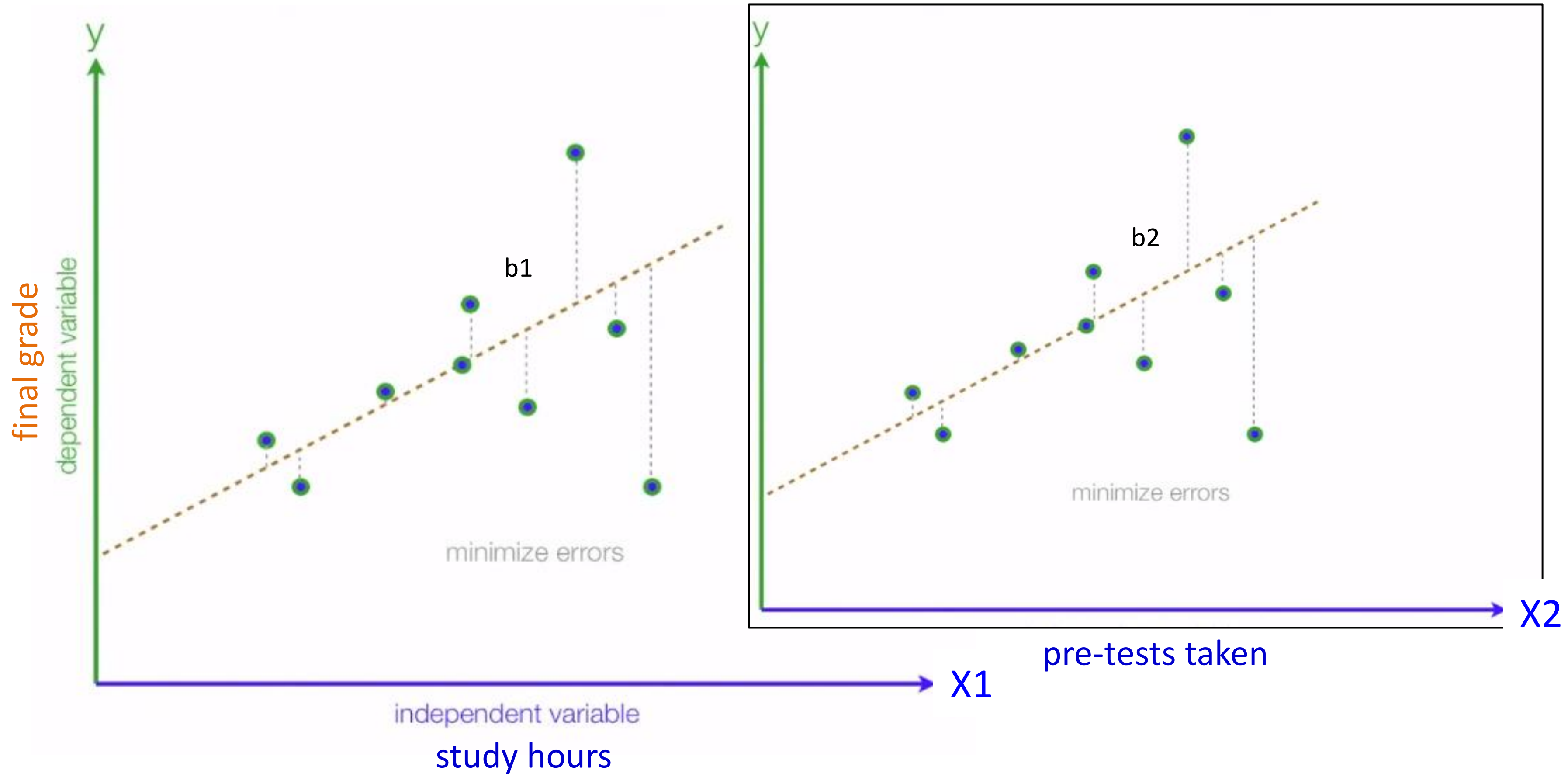


$$Y = b_0 + b_1X_1 + b_2X_2 + b_3X_3 \, .. \, b_nX_n$$

Multiple Linear Regression

Example: hours studied and pre-tests affect final score

$$y = b0 + b1*X1 + b2*X2 + E$$



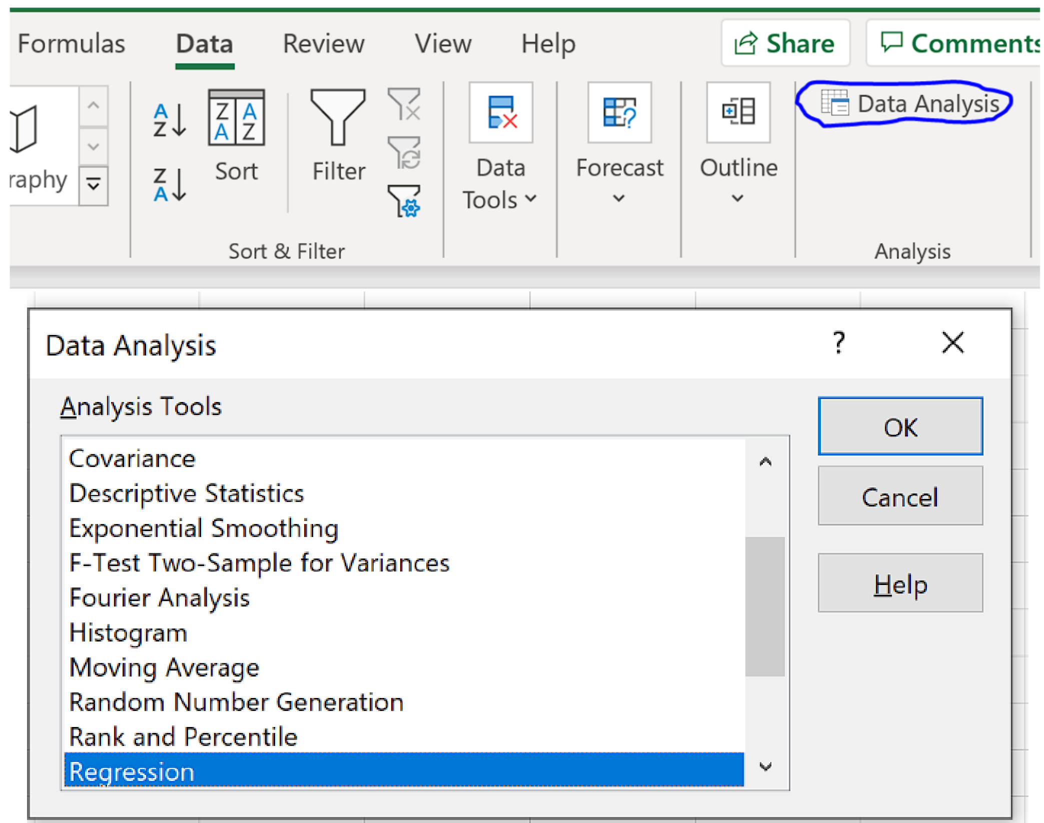
Multiple Linear Regression Example (1 of 2)

https://www.statology.org/multiple-linear-regression-excel/

- Hours studied and prep exams taken → exam score

	A	B	C
1	hours	prep_exams	score
2	1	1	76
3	2	3	78
4	2	3	85
5	4	5	88
6	2	2	72
7	1	2	69
8	5	1	94
9	4	1	94

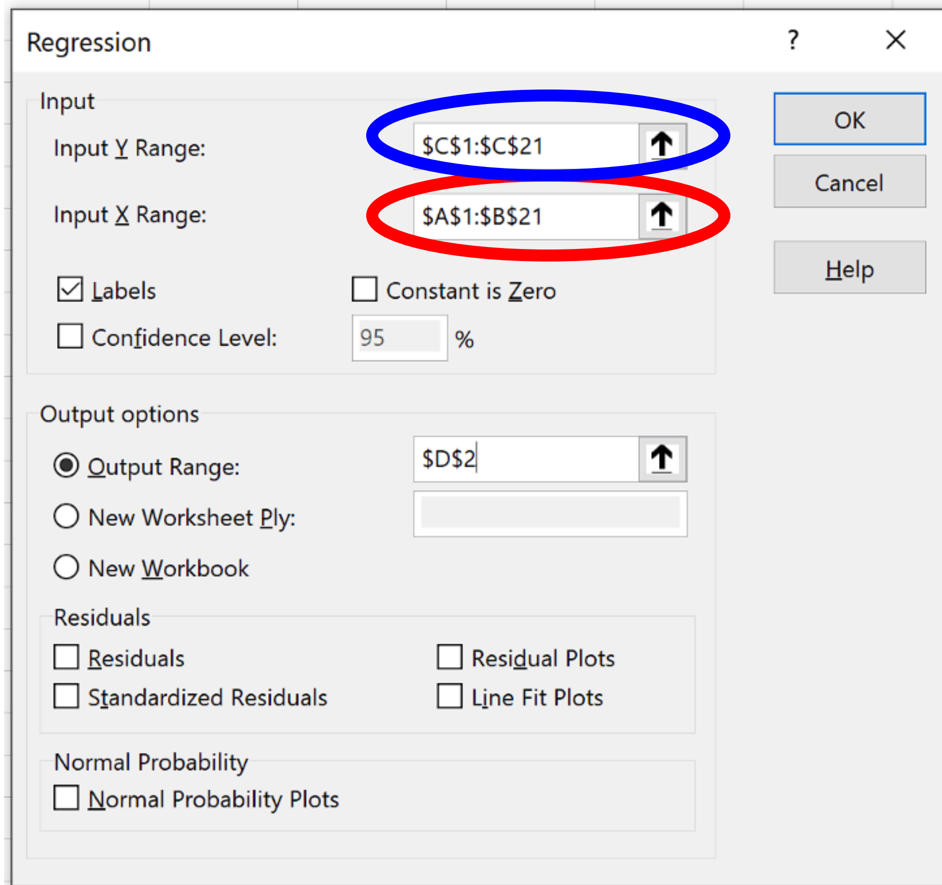
20 students



Multiple Linear Regression Example (2 of 2)

- Independent variable
- Covers both independent variables

	A	B	C
1	hours	prep_exams	score
2	1	1	76
3	2	3	78
4	2	3	85
5	4	5	88
6	2	2	72
7	1	2	69
8	5	1	94
9	4	1	94



Interpret

SUMMARY OUTPUT	
Regression Statistics	
Multiple R	0.857
R Square	0.734
Adjusted R Square	0.703
Standard Error	5.366
Observations	20

- R^2 0.734
- Overall significant ($p < 0.05$)
- Hours significant
- Prep exams not significant
- Base score without prep 67.67
- Each hour gains 5.56 percent

ANOVA

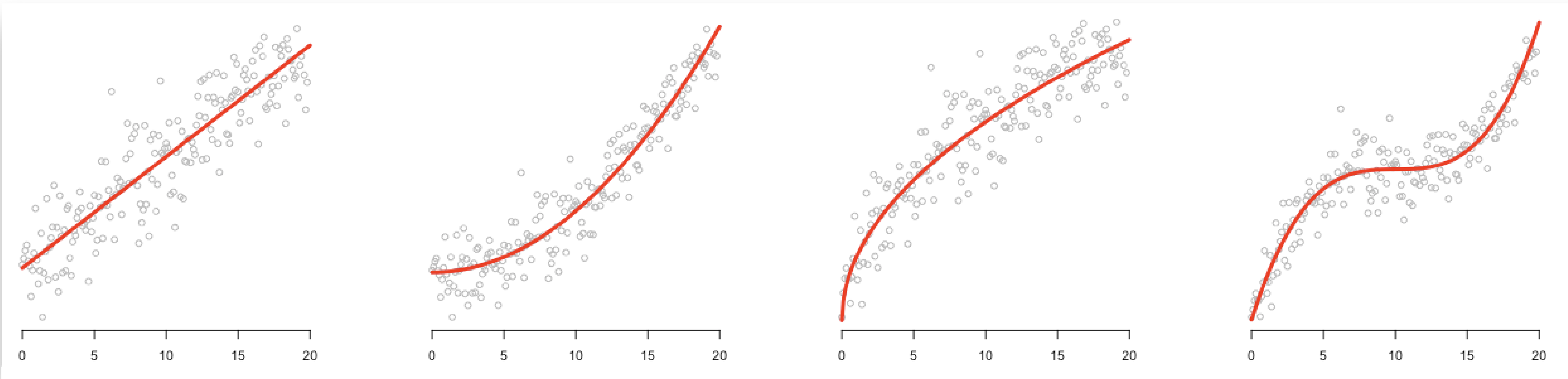
	df	*SS*	*MS*	*F*	*Significance F*
Regression	2	1350.76	675.38	23.46	0.00
Residual	17	489.44	28.79		
Total	19	1840.20			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	67.67	2.82	24.03	0.00	61.73	73.61
hours	5.56	0.90	6.18	0.00	3.66	7.45
prep_exams	-0.60	0.91	-0.66	0.52	-2.53	1.33

Score = 67.67 + 5.56 x hours – 0.60 x prep_exams

Beyond Linear Regression

https://medium.freecodecamp.org/learn-how-to-improve-your-linear-models-8294bfa8a731

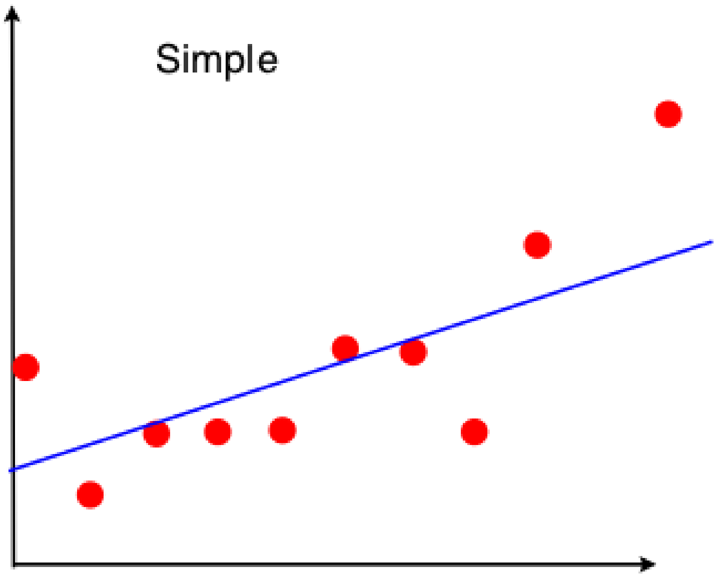


Linear Quadratic Root Cubic

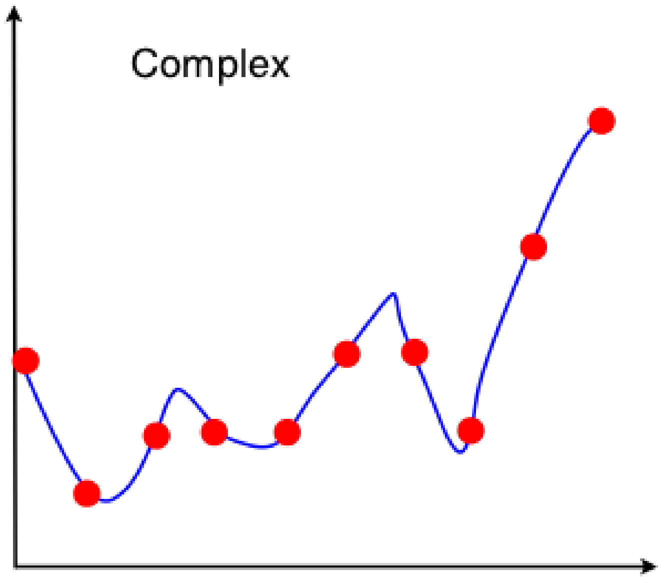
- More complex models – beyond just linear

$$Y = mX + b$$

More Complex Models



$y = 12x + 9$



$y = 18x^4 + 13x^3 - 9x^2 + 3x + 20$

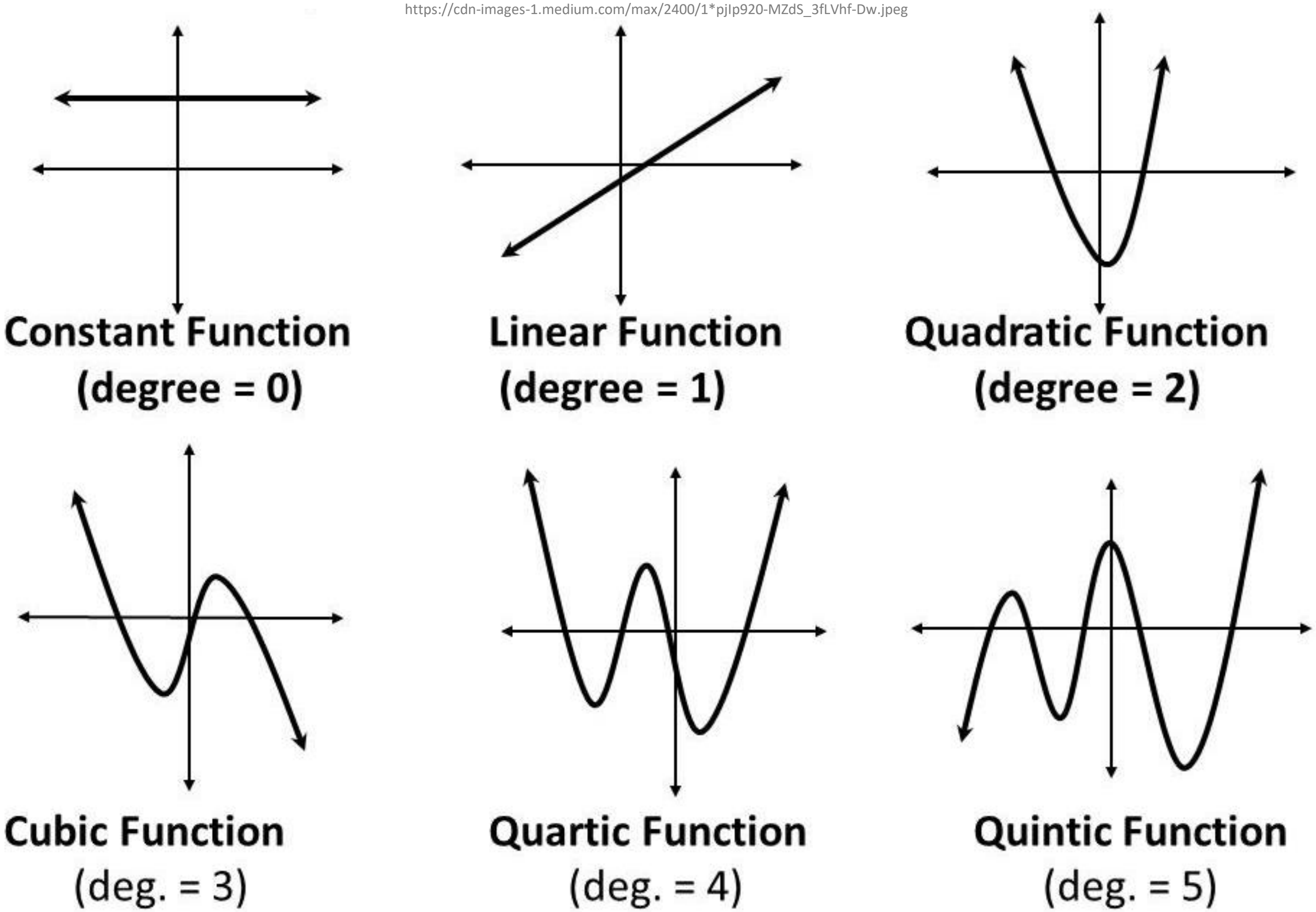
- Higher order polynomial model has less error

→ A "perfect" fit (no error)

- How does a polynomial do this?

Graphs of Polynomial Functions

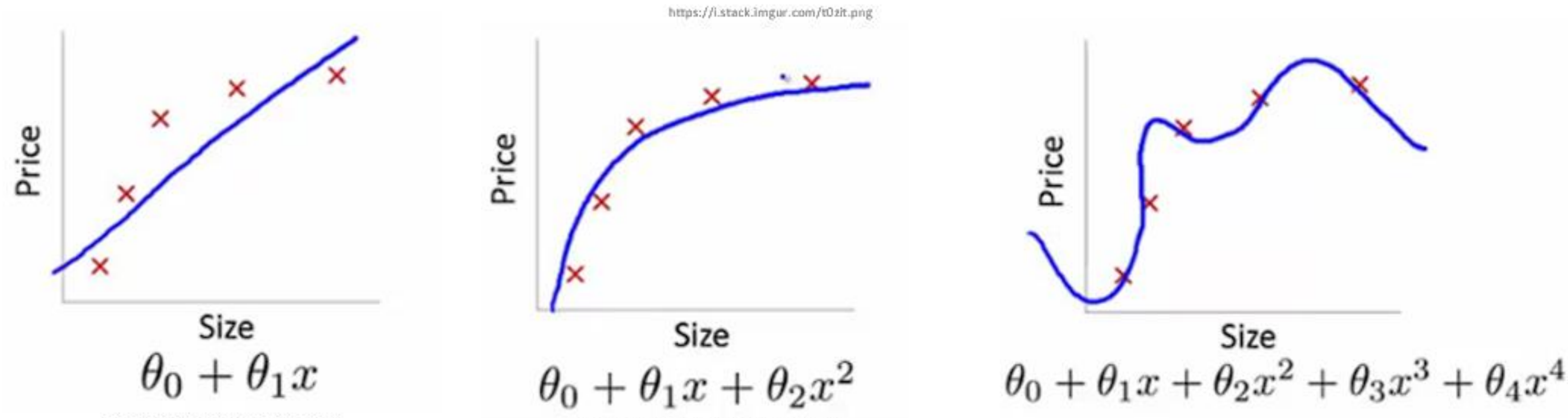
https://cdn-images-1.medium.com/max/2400/1*pjIp920-MZdS_3fLVhf-Dw.jpeg



Higher degree, more potential "wiggles"

But should you use?

Underfit and Overfit



Underfit | Just Right | Overfit

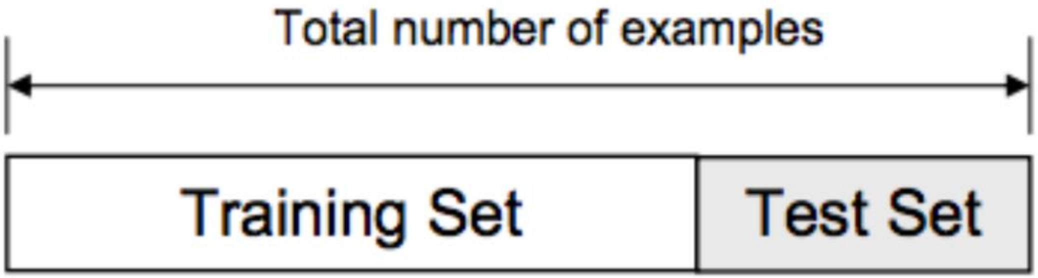
Test → Cross Validation

- Overfit analysis ameters than can be justified
- Underfit analysis does not adequately match data since parameters are missing

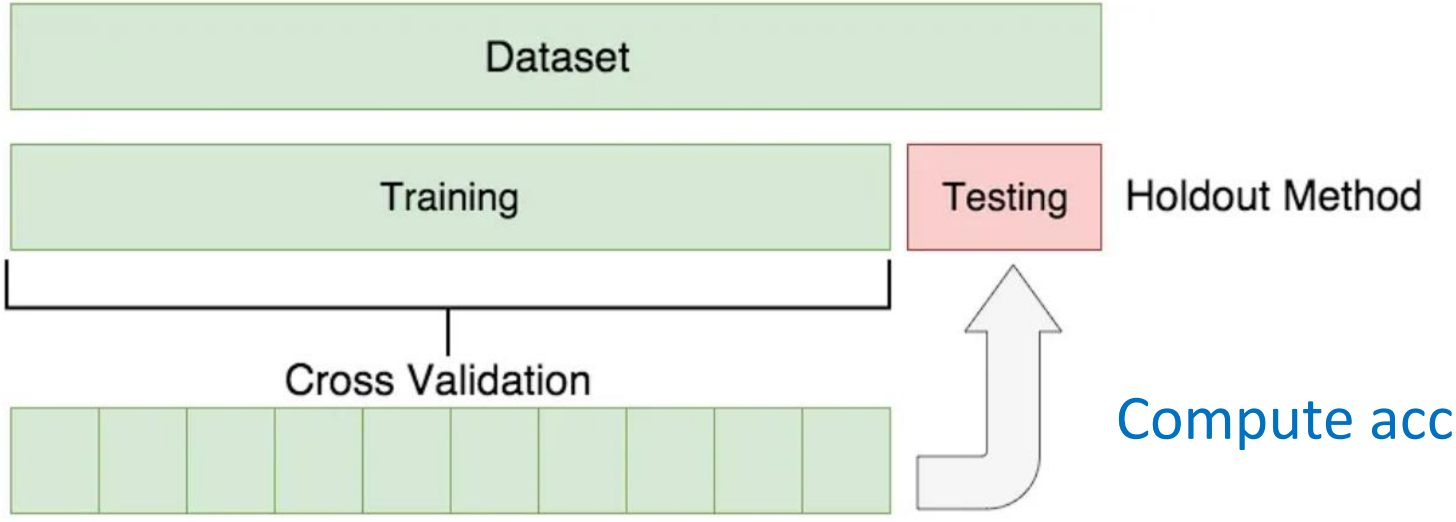
→ Both models fit well, but do not *predict* well (i.e., for non-observed values)

- Just right – fit data well “enough” with as few parameters as possible (*parsimonious* - desired level of prediction with as few terms as possible)

Cross Validation (1 of 2)



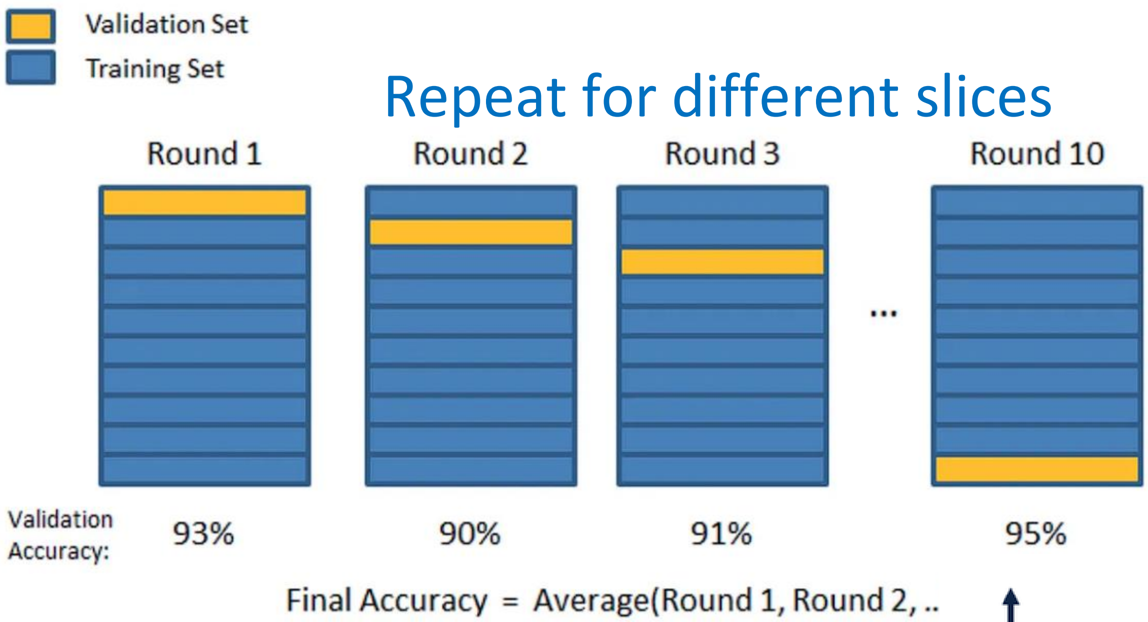
Use to build model



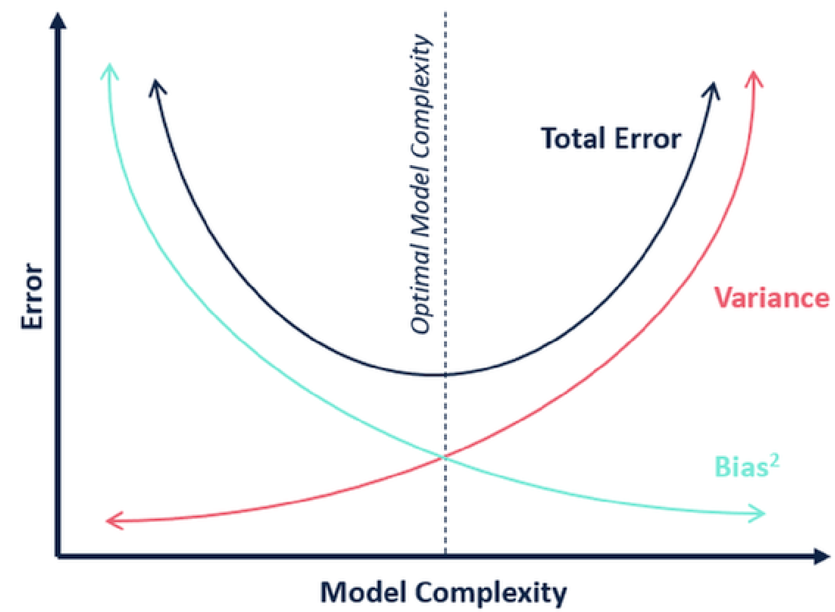
Compute accuracy

Cross Validation (2 of 2)

Repeat for different slices



- Overfit and Underfit will both have lower accuracy than "just right"



Summary

- Can use regression to predict un-measured values
- Before fit
  - Visual relationship (scatter plot) and residual analysis
- Strength of fit – R^2 and correlation (R)
- Beware
  - Correlation is not causation
  - Extrapolation
- Higher order, more complex models can fit better
  - Beware of overfit → less predictive power

https://d3h0owdjgzys62.cloudfront.net/images/3963/live_cover_art/thumb2x/summary_final.png