First Order Linear Difference Equations

Perhaps the simplest type of difference equation is the linear recurrence of the first order, namely,

$$x_{n+1} = a_n x_n + b_n, \quad n \geq m, \tag{1}$$

where both sequences $\{a_n\}_{n\geq m}$, $\{b_n\}_{n\geq m}$, and the initial element x_m are known.

The homogeneous recurrence (when $b_n = 0$) is very simple to handle: we investigate it by iteration. Since the relation $x_{n+1} = a_n x_n$ holds for all $n \geq m$, then we can iterate it:

$$x_{m+1} = a_m x_m, \quad x_{m+2} = a_{m+1}x_{m+1} = a_{m+1}a_m x_m, \quad x_{m+3} = a_{m+2}x_{m+2} = a_{m+2}a_{m+1}a_m x_m \ldots \tag{$\star$}$$

This suggests a *guess solution* as follows

$$x_n = x_m \prod_{m\leq k<n} a_k.$$

On substitution we find it clearly satisfies the recurrence $x_{n+1} = a_n x_n$, which proves it correctness.

Note: This approach for solution, iteration-guess-proof (by substitution), is a very common method of handling recurrences. The similarity to the process of proof by mathematical induction is evident. A variant is as follows: instead of starting at the lowest end of the index range, we begin at some arbitrary point, say, at n, and unravel the recurrence by going down the range. Thus we would write instead of ($\star$) above

$$x_{n+1} = a_n x_n = a_n(a_{n-1}x_{n-1}) = a_n a_{n-1}(a_{n-2}x_{n-2}) = a_n a_{n-1}a_{n-2}(a_{n-3}x_{n-3})\ldots \tag{$\star\star$}$$

again followed by a guess and a proof by substitution. Which direction to take? The one that fits the problem best. In this case: the first seems to me more natural.

The non-homogeneous recurrence can be solved similarly by unraveling it:

$$\begin{aligned}
x_{n+1} &= b_n + a_n(a_{n-1}x_{n-1} + b_{n-1}) = b_n + a_n b_{n-1} + a_n a_{n-1} x_{n-1} \\
&= b_n + a_n b_{n-1} + a_n a_{n-1}(a_{n-2}x_{n-2} + b_{n-2}) \\
&= b_n + a_n b_{n-1} + a_n a_{n-1} b_{n-2} + a_n a_{n-1} a_{n-2} x_{n-2} \\
&= b_n + a_n b_{n-1} + a_n a_{n-1} b_{n-2} + a_n a_{n-1} a_{n-2} b_{n-3} + a_n a_{n-1} a_{n-2} a_{n-3} x_{n-3} \\
\ldots & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
&= b_n + a_n b_{n-1} + a_n a_{n-1} b_{n-2} + \cdots + a_n a_{n-1} \cdots a_{m+1}(a_m x_m + b_m).
\end{aligned}$$

We obtained the guess solution of Eq. (1) to be

$$x_{n+1} = \sum_{j=m}^{n} b_j \prod_{k=j+1}^{n} a_k + x_m \prod_{k=m}^{n} a_k. \tag{2}$$

This guess still needs to be checked by substitution, to verify our guess — and this is a straightforward calculation. Note that it is a sum of the general solution for the homogeneous equation we saw before, and a particular solution of the non-homogeneous one for the case $x_m = 0$.