First Order Linear Difference Equations

Perhaps the simplest type of difference equation is the linear recurrence of the first order, namely,

$$x_{n+1} = a_n x_n + b_n, \quad n \geq m,$$  \hspace{1cm} (1)

where both sequences \( \{a_n\}_{n \geq m} \), \( \{b_n\}_{n \geq m} \), and the initial element \( x_m \) are known.

The homogeneous recurrence (when \( b_n = 0 \)) is very simple to handle: we investigate it by iteration. Since the relation \( x_{n+1} = a_n x_n \) holds for all \( n \geq m \), then we can iterate it:

$$x_{m+1} = a_m x_m, \quad x_{m+2} = a_{m+1} x_{m+1} = a_{m+1} a_m x_m, \quad x_{m+3} = a_{m+2} x_{m+2} = a_{m+2} a_{m+1} a_m x_m \ldots$$  \hspace{1cm} (*)

This suggests a guess solution as follows

$$x_n = x_m \prod_{m \leq k < n} a_k.$$  

On substitution we find it clearly satisfies the recurrence \( x_{n+1} = a_n x_n \), which proves it correctness.

Note: This approach for solution, iteration-guess-proof (by substitution), is a very common method of handling recurrences. The similarity to the process of proof by mathematical induction is evident. A variant is as follows: instead of starting at the lowest end of the index range, we begin at some arbitrary point, say, at \( n \), and unravel the recurrence by going down the range. Thus we would write instead of (\( * \)) above

$$x_{n+1} = a_n x_n = a_n (a_{n-1} x_{n-1}) = a_n a_{n-1} (a_{n-2} x_{n-2}) = a_n a_{n-1} a_{n-2} (a_{n-3} x_{n-3}) \ldots$$  \hspace{1cm} (**)

again followed by a guess and a proof by substitution. Which direction to take? The one that fits the problem best. In this case: the first seems to me more natural.

The non-homogeneous recurrence can be solved similarly by unraveling it:

$$x_{n+1} = b_n + a_n (a_{n-1} x_{n-1} + b_{n-1}) = b_n + a_n b_{n-1} + a_n a_{n-1} x_{n-1}$$

$$= b_n + a_n b_{n-1} + a_n a_{n-1} (a_{n-2} x_{n-2} + b_{n-2})$$

$$= b_n + a_n b_{n-1} + a_n a_{n-1} b_{n-2} + a_n a_{n-1} a_{n-2} x_{n-2}$$

$$= b_n + a_n b_{n-1} + a_n a_{n-1} b_{n-2} + a_n a_{n-1} a_{n-2} b_{n-3} + a_n a_{n-1} a_{n-2} a_{n-3} x_{n-3}$$

$$\ldots$$

$$= b_n + a_n b_{n-1} + a_n a_{n-1} b_{n-2} + \ldots + a_n a_{n-1} \ldots a_{m+1} (a_m x_m + b_m).$$

We obtained the guess solution of Eq. (1) to be

$$x_{n+1} = \sum_{j=m}^{n} b_j \prod_{j+1 \leq k \leq n} a_k + x_m \prod_{k=m}^{n} a_k. \hspace{1cm} (2)$$
This guess still needs to be checked by substitution, to verify our guess — and this is a straightforward calculation. Note that it is a sum of the general solution for the homogeneous equation we saw before, and a particular solution of the non-homogeneous one for the case $x_m = 0$. 