0.0.1 Multisection of Series

Given a sequence $\{a_k\}_{k\geqslant 0}$ and its generating function $a(z) = \sum_{k\geqslant 0} a_k z^k$, we are asked to compute the sum of every second term: $\sum_{k\geqslant 0} a_{2k}$. Similar situations seem to crop up often enough in analysis of algorithms to deserve a unified approach. The following computation gives what we need:

$$\frac{1}{2}[a(1)+a(-1)] = \frac{1}{2}\left[\sum_{k\geqslant 0} a_k + \sum_{k\geqslant 0} a_k(-1)^k\right] = \frac{1}{2}\sum_{k\geqslant 0} a_k\left(1+(-1)^k\right) = \sum_{k\geqslant 0} a_{2k}, \qquad (1)$$

since only the even terms survive. For example, what is $\sum_k \binom{n}{2k}$? With the above idea, $a_k = \binom{n}{k}$, and then $a(z) = \sum_k \binom{n}{k} z^k = (1+z)^n$. We see that $a(1) = 2^n$, and $a(-1) = \delta_{0,n}$, and therefore, $\sum_k \binom{n}{2k} = (2^n + \delta_{0,n})/2$. Not a surprising result, really.

The same idea holds when we need the sum of every rth term, for any $r \in \mathbb{I}_2$. The calculation now calls for the GF of the sequence evaluated at the unit roots of order r.

The unit roots of order r are the r numbers $\omega_{k,r} = e^{2i\pi k/r}, \quad k = 0, 1, \ldots, r-1$. We call $\omega_{1,r}$ the *primitive* unit root of order r. It is a root of 1, since $\omega_{1,r}^r = \left(e^{2i\pi/r}\right)^r = e^{2i\pi} = 1$, and it is called the primitive one since the kth power of $\omega_{1,r}$ equals $\omega_{k,r}$; hence its powers generate all the other $r-1$ unit roots of order r.

Note: You may think that this is less than a perfect reason for the term 'primitive,' since it is possible to find other, non-primitive roots that have the same property (although they generate the other roots in a different order!). For example, $u = e^{2\times 2i\pi/3}$ and its powers u^2 and u^3 are the three roots of order three. Very true, but notation and definitions aim at naturalness, not artificial uniqueness.

Let us keep r fixed, and $\omega \stackrel{\text{def}}{=} \omega_{1,r}$. Such roots of unity satisfy the following "circular" (or orthogonal) relation:

$$\sum_{j=0}^{r-1} (\omega^k)^j = \begin{cases} r & \text{when } k \text{ is a multiple of } r \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

The reason is easy to see. Let k be any positive integer, then

$$\sum_{j=0}^{r-1} (\omega^k)^j = \frac{1-(\omega^k)^r}{1-\omega^k} = \frac{1-\omega^{kr}}{1-\omega^k}.$$

The numerator is $1-\omega^{kr} = 1-(\omega^r)^k = 1-1 = 0$ for any integer k; unless k is a multiple of r the denominator is nonzero, and the right-hand side above vanishes. However, when $k = mr$ for some integer m, the right-hand side has the value 0/0, and we use L'Hôpital's rule to evaluate it, by differentiating both numerator and denominator (with respect to ω), to get

$$\frac{-kr\omega^{kr-1}}{-k\omega^{k-1}} = r\omega^{k(r-1)} = r\omega^{mr(r-1)} = r\left(\omega^r\right)^{m(r-1)} = r.$$

We use this result in the following calculation:

$$\frac{1}{r}\sum_{j=0}^{r-1} a(\omega^j) = \frac{1}{r}\sum_{j=0}^{r-1}\sum_{k\geqslant 0} a_k\omega^{jk} = \frac{1}{r}\sum_{k\geqslant 0} a_k \sum_{j=0}^{r-1}(\omega^k)^j = \frac{1}{r}\sum_{k\geqslant 0} a_k \times r[k = mr \mid m \in \mathbb{N}] = \sum_{m\geqslant 0} a_{mr}. \qquad (3)$$

Such a sum is called *multisection* of order r.

Example 1

Compute $\sum_k \binom{n}{3k}$. We pick the term $a_k = \binom{n}{k} \implies a(z) = \sum_k \binom{n}{k} z^k = (1+z)^n$. We showed above that

$$\sum_k \binom{n}{3k} = \frac{1}{3} \sum_{j=0}^{2} a(\omega^j),$$

with ω here be the cubic root of one:

$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2},$$

and substitution provides a less-than-obvious expression:

$$\sum_k \binom{n}{3k} = \frac{1}{3}\left(2^n + (1+\omega)^n + (1+\omega^2)^n\right) = \frac{1}{3}\left(2^n + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^n + \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^n\right).$$

However, since $\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = e^{\pm i\pi/3}$, we use this to get

$$\sum_k \binom{n}{3k} = \frac{1}{3}\left(2^n + e^{in\pi/3} + e^{-in\pi/3}\right) = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right).$$

Hence the multisection is essentially a third of the full sum, with a small oscillation.

Example 2

To find the sum $\sum_{k\geqslant 0} \frac{1}{(3k)!}$:

We use the OGF for the sequence $\{\frac{1}{k!}\}$, $\sum_{k\geqslant 0} \frac{x^n}{n!} = e^x$. Let $\omega = e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ be the primitive cubic root of 1, as obtained above. Using the formula in Eq. (1.3), we obtain the value of the series:

$$\sum_{k\geqslant 0} \frac{1}{(3k)!} = \frac{1}{3}\left[e^1 + e^{\omega} + e^{\omega^2}\right] = \frac{1}{3}\left[e + e^{-1/2}\left(e^{i\sqrt{3}/2} + e^{-i\sqrt{3}/2}\right)\right] = \frac{1}{3}\left[e + \frac{2}{\sqrt{e}}\cos\left(\frac{\sqrt{3}}{2}\right)\right].$$

with the numerical value 1.168058313....

Example 3

To find the sum $S = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{2k+1}\binom{n}{2k}$.

Let

$$\begin{aligned} a(x) &= \sum_k \frac{1}{k+1}\binom{n}{k} x^k = \sum_k \frac{1}{n+1}\binom{n+1}{k+1} x^k \\ &= \frac{1}{n+1}\sum_{k=1}^{n+1}\binom{n+1}{j} x^{j-1} = \frac{x^{-1}}{n+1}\left[(1+x)^{n+1} - 1\right]. \end{aligned}$$

Therefore $S = \dfrac{a(1)+a(-1)}{2} = \dfrac{2^n}{n+1}$.

Example 4

Calculate $\sum_{n\geqslant 0} y^{5n}$. Surely you can do this geometrical sum directly. Show how to recover the same result when viewing this as a 5-way multisection.

Let $a_n = y^n$, $n \geqslant 0$. The OGF is $a(z) = \sum_{n\geqslant 0} a_n z^n = \sum_{n\geqslant 0} y^n z^n = 1/(1-yz)$. Let $\omega = e^{2\pi i/5}$ be the primitive fifth root of 1. We get all fifth roots of 1 as $\{\omega^0 = 1,\ \omega,\ \omega^2,\ \omega^3,\ \omega^4\}$. From Eq. (1.3) we write, ending with a common denominator:

$$\sum_{n\geqslant 0} y^{5n} = \frac{1}{5}[a(1)+a(\omega)+a(\omega^2)+a(\omega^3)+a(\omega^4)] = \frac{1}{5}\sum_{j=0}^{4}\frac{1}{1-y\omega^j} = \frac{1}{5}\frac{\mathrm{N}}{\mathrm{D}}.$$

This is the kind of calculation one should relegate to MAPLE, but it behooves us to see the internals:

$$\begin{aligned}
\mathrm{N} &= (1-y\omega)(1-y\omega^2)(1-y\omega^3)(1-y\omega^4)+(1-y)(1-y\omega^2)(1-y\omega^3)(1-y\omega^4)\\
&\quad +(1-y)(1-y\omega)(1-y\omega^3)(1-y\omega^4)+(1-y)(1-y\omega)(1-y\omega^2)(1-y\omega^4)\\
&\quad +(1-y)(1-y\omega)(1-y\omega^2)(1-y\omega^3)\\
&= 5-4y\sum_{k=0}^{4}\omega^k+3y^2[\omega+\omega^2+2\omega^3+2\omega^4+2\omega^5+\omega^6+\omega^7]\\
&\quad -2y^3[\omega^3+\omega^4+2\omega^5+2\omega^6+2\omega^7+\omega^8+\omega^9]+y^4[\omega^6+\omega^7+\omega^8+\omega^9+\omega^{10}].
\end{aligned}$$

$$\begin{aligned}
\text{and}\quad \mathrm{D} &= (1-y)(1-y\omega)(1-y\omega^2)(1-y\omega^3)(1-y\omega^4)\\
&= 1-y\sum_{k=0}^{4}\omega^k+y^2[\omega+\omega^2+2\omega^3+2\omega^4+2\omega^5+\omega^6+\omega^7]\\
&\quad -y^3[\omega^3+\omega^4+2\omega^5+2\omega^6+2\omega^7+\omega^8+\omega^9]+y^4\sum_{k=6}^{10}\omega^k-y^5\omega^{10}.
\end{aligned}$$

We simplify, based on Eq. (1.2) and $\omega^{5+s} = \omega^s$,

$$\mathrm{N} = 5-4y\sum_{k=0}^{4}\omega^k+6y^2\sum_{k=0}^{4}\omega^k-4y^3\sum_{k=0}^{4}\omega^k+y^4\sum_{k=0}^{4}\omega^k = 5;$$

$$\mathrm{D} = 1-y\sum_{k=0}^{4}\omega^k+2y^2\sum_{k=0}^{4}\omega^k-2y^3\sum_{k=0}^{4}\omega^k+y^4\sum_{k=6}^{10}\omega^k-y^5\omega^{10} = 1-y^5.$$