Theorem 1 *The probability that a needle of length l will fall across a crack on a floor made up of planks of width $2l$ is $1/\pi$. The cracks between the planks are assumed to be of width 0.*

Proof:

The proof is not particularly rigorous but works. We will consider only one crack between two planks since it won't change the result. Consider Figure 1, in which the thick line is the needle and we consider only the case in which the needle falls across the upper crack.

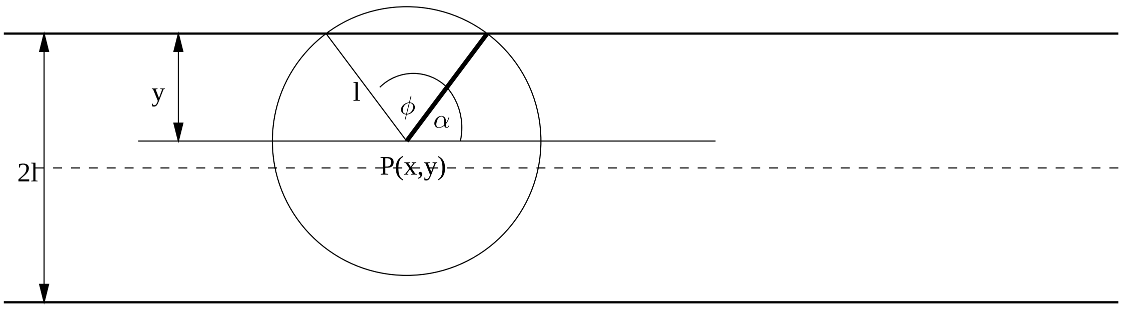

Figure 1: Buffon's Needles

The needle falls at a random location, i.e., $P(x, y)$ has uniformly distributed coordinates x and y. Since the planks are infinitely long, we do not need to consider the x-coordinate. As for y, we will consider only the case in which the distance between P and the upper crack in the figure is not bigger than l since otherwise the needle can not fall across the crack. We do not need to consider the other case, in which the needle falls in the lower half of a plank, since it is a mirror-image of our case, and therefore the probability will be the same as for our case. Also, those events are independent from each other, since the needle can never fall across two cracks at the same time.

Note that the needle falls at a random location *and* at a random angle α. We measure the y-coordinate as distance from the crack. Thus, y is uniformly distributed between 0 and l, while α is uniformly distributed between 0 and 2π. To simplify the proof, we will set $l = 1$ without loss of generality.

Now, the probability that the needle falls across the crack for a certain y is $p(y) = \frac{\phi}{2\pi}$. It is clear from the picture that $\phi = \pi - 2\alpha$ and that $\sin\alpha = y$ for all $y \in [0, 1]$. Therefore,

$$p(y) = \frac{\pi - 2\arcsin y}{2\pi} = \frac{1}{2} - \frac{\arcsin y}{\pi}.$$

Considering that y is uniformly distributed between 0 and 1, we can now simply find the integral for $p(y)$ in the interval $[0, 1]$ to get the probability p. Note that $\int \arcsin x \, dx = x \arcsin x + \sqrt{1 - x^2} + C$.

$$\begin{aligned}
p &= \int_0^1 p(y)dy \\
&= \int_0^1 \frac{1}{2} - \frac{\arcsin y}{\pi}\, dy \\
&= \left.\frac{x}{2} - \frac{x \arcsin x + \sqrt{1-x^2}}{\pi}\right|_0^1 \\
&= \left(\frac{1}{2} - \frac{\frac{\pi}{2}}{\pi}\right) - \left(0 - \frac{1}{\pi}\right) \\
&= \frac{1}{2} - \frac{1}{2} + \frac{1}{\pi} \\
&= \frac{1}{\pi} \qquad \textbf{q.e.d.}
\end{aligned}$$