

$$\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k}; \quad (\mathbf{b}) \binom{r}{k} = \frac{r}{k} \binom{r-1}{k-1} = \frac{r-k+1}{k} \binom{r}{k-1} = \frac{r}{r-k} \binom{r-1}{k}. \quad (1.1)$$

$$\binom{r}{k} = \binom{r}{r-k}; \quad (\mathbf{b}) \binom{r}{k, m-k, r-m} = \binom{r}{m} \binom{m}{k} = \binom{r}{k} \binom{r-k}{m-k} = \binom{r}{m-k} \binom{r-m+k}{k}, \quad 0 \leq k \leq r \in \mathbb{Z}. \quad (1.2)$$

$$\binom{-r}{k} (-1)^k = \binom{r+k-1}{k}; \quad (\mathbf{b}) \binom{k+1/2}{n} = \begin{cases} 4^{-n} \binom{2n}{n} (-1)^{n+k-1} \binom{n}{k+1} \binom{2n}{2k+2}^{-1} & k < n \\ 4^{-n} \binom{2n}{n} \binom{2k+1}{2n} \binom{k}{n}^{-1} & k \geq n \end{cases} \quad (1.3)$$

$$\sum_{k=0}^n \binom{r+k}{r} = \binom{r+n+1}{n}; \quad (\mathbf{b}) \sum_{k=m}^n \binom{r}{k} (-1)^k = \binom{r-1}{m-1} (-1)^m + \binom{r-1}{n} (-1)^n; \quad (1.4)$$

$$\sum_{k=0}^m \binom{k+n}{t} = \binom{m+n+1}{t+1} - \binom{n}{t+1} \Rightarrow (\mathbf{b}) \sum_{k=1}^n \binom{a-k}{t} = \binom{a}{t+1} - \binom{a-n}{t+1}, \quad a-t \geq n. \quad (1.5)$$

$$\sum_{k=m}^n \binom{k}{m} = \binom{n+1}{m+1}; \quad (\mathbf{b}) \sum_{k=0}^m k \binom{k+n}{k} = (n+1) \binom{m+n+1}{m-1}; \quad (\mathbf{c}) \sum_{k=0}^n \binom{n-k}{k} = \sum_{k \geq 0} \binom{k}{n-k} = F_{n+1}. \quad (1.6)$$

$$\sum_{k=1}^n k! \cdot k = (n+1)! - 1; \quad (\mathbf{b}) \sum_{k=1}^n 2k(2k-1)!! = (2n+1)!! - 1; \quad (\mathbf{c}) \sum_{k=0}^n (2k+1)(2k)!! = (2n+2)!! - 2. \quad (1.7)$$

$$\sum_{k=a}^n k^m = \frac{k^{m+1}}{m+1} \Big|_{k=a}^{n+1}, \quad m \neq -1, a \geq m; \quad (\mathbf{b}) \sum_k \binom{n}{k}^2 = \binom{2n}{n}; \quad (\mathbf{c}) \sum_k k \binom{n}{k}^2 = (2n-1) \binom{2n-2}{n-1}; \quad (1.8)$$

$$\sum_{k=0}^n \binom{x+k}{k}^{-1} = \frac{x}{x-1} \left[ 1 - \binom{x+n}{n+1}^{-1} \right]; \quad (\mathbf{b}) \sum_{k=0}^n (-1)^k \binom{x}{k}^{-1} = \frac{x+1}{x+2} + \frac{x-n}{x+2} \frac{(-1)^n}{\binom{x}{n+1}}. \quad (\mathbf{c}) \sum_{k=0}^n \frac{4^k}{\binom{2k}{k}} = \frac{1}{3} + \frac{2n+1}{3} \frac{4^{n+1}}{\binom{2n+1}{n+1}}. \quad (1.9)$$

$$\sum_{k=0}^n \binom{k+u}{k} z^k = \frac{1}{u!} \frac{d^u}{dz^u} z^u \frac{1-z^{n+1}}{1-z} = \frac{1-z^{n+1} \sum_{i=0}^u \binom{i+n}{i} (i-z)^i}{(1-z)^{u+1}}, \quad u \in \mathbb{N}; \quad (\mathbf{b}) \sum_{k=0}^n \binom{1/2}{k} (-1)^k = \binom{-1/2}{n} (-1)^n. \quad (1.10)$$

$$\sum_{k=0}^m \binom{k+n-m-1}{k} z^k = \sum_{k=0}^m \binom{n}{k} z^k (1-z)^{m-k}, \quad m \leq n \in \mathbb{N}; \quad (\mathbf{b}) \sum_{k=0}^n \binom{n-k}{k} z^k = \frac{(1+s)^{n+1} - (1-s)^{n+1}}{2^{n+1} s}, \quad s = \sqrt{1+4z}. \quad (1.11)$$

$$\sum_{k \geq 0} \binom{k+u}{k} z^k = \frac{1}{(1-z)^{u+1}}; \quad (\mathbf{b}) \sum_{k \geq 0} \binom{s+k}{u} z^k = \frac{z^{u-s}}{(1-z)^{u+1}} - [u < s] \sum_{k=0}^{s-u-1} \binom{u+k}{u} z^{k-(s-u)} \quad u, s \in \mathbb{N}; \quad |z| < 1. \quad (1.12)$$

$$\sum_{k=0}^n \binom{x}{k} \frac{k \cdot k!}{x^{k+1}} = 1 - \binom{x}{n+1} \frac{(n+1)!}{x^{n+1}}; \quad (\mathbf{b}) \sum_{k \geq 1} \frac{x^k}{k \cdot k!} = E_i(x) - \gamma - \ln x = \frac{e^x}{x} (1 + O(x^{-1})). \quad (1.13)$$

$$\sum_{k \geq 0} \binom{r-tk}{k} z^k = \frac{x^{r+1}}{(t+1)x-t}, \quad (\mathbf{b}) \sum_{k \geq 0} \binom{r-tk}{k} \frac{r}{r-tk} z^k = x^r, \quad \text{where } z = x^{t+1} - x^t, \quad (1.14)$$

and the solution is chosen be regular in  $z \in \{|z| < |t^t(t+1)^{-(t+1)}|\}$ . This is also the region of convergence.

$$\sum_{k=1}^n \frac{1}{k} \binom{r+n-k}{r} = \binom{r+n}{n} (H_{n+r} - H_r). \quad (\mathbf{b}) \sum_{k=0}^n \binom{n+k}{k} 2^{-k} = 2^n; \quad (\mathbf{c}) \sum_{k=0}^n \binom{2k}{k} 4^{-k} = (2n+1) \binom{2k}{k} 4^{-n}; \quad (1.15)$$

$$\sum_{k \geq 1} \binom{n}{k} \frac{(-1)^{k+1}}{k} (x+ky)^r = x^r H_n + r x^{r-1} y, \quad 0 \leq r \leq n. \quad (\mathbf{b}) \sum_{j=0}^n \binom{n}{j} \frac{(-1)^j}{x+j} = \frac{1}{x \binom{n+x}{n}}, \quad x \notin 0, -1, \dots, -n. \quad (1.16)$$

$$\sum_k \binom{n}{k} (-1)^{n-k} k^n = n! \quad (\mathbf{b}) \sum_k \binom{n}{k} (-1)^{n-k} k^{n+1} = \frac{n}{2} (n+1)! \quad (\mathbf{c}) \sum_k \binom{n}{k} (tk+r)^{k-1} (t(n-k)+s)^{n-k} = \frac{(tn+r+s)^n}{r} \quad (1.17)$$

$$c_{m,n} \equiv \sum_k \binom{n}{k} (a+bk)^m (-1)^k = n! (-1)^n \sum_i \binom{m}{i} \{n\}_i^i a^{m-i} b^i [= 0 \mid m < n]; \quad (\mathbf{b}) \sum_{m \geq n} c_{m,n} = \frac{1/(1-a)}{\binom{n-(1-a)/b}{n}} := \frac{\delta(1, a-b)}{(1-a)(n+1)} \quad (1.18)$$

needs  $|a+bk| < 1, k \in [n]$

$$\sum_{k=0}^{r-1} \binom{n+1}{k} (r-k)^n (-1)^k = \sum_{k \geq 0} \binom{n-k}{r-1} \{k\} (-1)^{k+n+r+1} = A(n, r) = \langle n_{r-1} \rangle \text{ Eulerian \#s.} \quad (1.19)$$

$$\sum_{k \geq 0} \binom{2k}{k} z^k = \frac{1}{s}, \quad (\mathbf{b}) \sum_{k \geq 0} \binom{2k}{k} k z^k = \frac{2z}{s^3}, \quad (\mathbf{c}) \sum_{k \geq 0} \binom{2k}{k} \frac{z^k}{k+1} = \frac{1-s}{2z}, \quad (\mathbf{d}) \sum_{k \geq 0} \binom{2k+1}{k} z^k = \frac{1-s}{2zs}, \quad s = \sqrt{1-4z}. \quad (1.20)$$

$$\sum_{k \geq 0} \binom{2k}{k} \frac{z^{2k+1}}{2k+1} = \frac{\sin^{(-1)}(2z)}{2}, \quad (\mathbf{b}) \sum_{k \geq 0} \binom{2k}{k} \frac{z^k}{2k-1} = -s, \quad (\mathbf{c}) \sum_{k \geq 1} \binom{2k}{k} \frac{z^k}{k} = \log \frac{1-s}{z(1+s)}, \quad \text{for } s \text{ see (1.20)} \quad (1.21)$$

$$\sum_{k \geq 0} \binom{2k+a}{k} x^k = \frac{1}{s} \left( \frac{1-s}{2x} \right)^a a \in \mathbb{C}, \quad (\mathbf{b}) \sum_{k \geq 0} z^k \sum_{r=0}^{\lfloor k/2 \rfloor} \binom{k}{r} x^r = \frac{1/2}{1-z(1+x)} \left( 1 + \frac{1-2xz}{\sqrt{1-4xz^2}} \right) \quad \text{When bisected:} \quad (1.22)$$

$$\sum_{l \geq 0} z^l \sum_{r=0}^l \binom{2l+1}{r} x^r = \frac{1/2}{1-z(1+x)^2} \left( 1+x + \frac{1-x}{\sqrt{1-4xz}} \right); \quad (\mathbf{b}) \sum_{l \geq 0} z^l \sum_{r=0}^l \binom{2l}{r} x^r = \frac{1/2}{1-z(1+x)^2} \left( 1 + \frac{1-2x(1+x)z}{\sqrt{1-4xz}} \right) \quad (1.23)$$

$$\Gamma(n+1) = n! = \sqrt{2\pi n} n^{n+1/2} e^{-n} \left( 1 + \frac{1}{12n} + \frac{1}{288n^2} + O(n^{-3}) \right) \quad n!! = 2^{\lfloor n/2 \rfloor} \frac{\Gamma(1+n/2)}{\pi^{n/2 - \lfloor n/2 \rfloor}} \quad \Gamma\left(\frac{2k+1}{2}\right) = \frac{\sqrt{\pi}(2k)!}{4^k k!} \quad (2.1)$$

$$\frac{\Gamma(b+n+1)}{\Gamma(b)} = (b+n)^{n+1}; \quad \frac{\Gamma(-n)}{\Gamma(-m)} = \frac{\Gamma(m+1)}{\Gamma(n+1)} (-1)^{n-m}, \quad m, n \in \mathbb{N}. \quad \frac{\Gamma(n+a)}{\Gamma(n+b)} = n^{a-b} \left[ 1 + \frac{(a-b)(a+b-1)}{2n} + O(n^{-2}) \right] \quad (2.2)$$

$$\binom{1/2}{k} = \frac{1}{1-2k} \binom{-1/2}{k}; \quad \binom{n-1/2}{n} = (-1)^n \binom{-1/2}{n} = 4^{-n} \binom{2n}{n} \approx \frac{1}{\sqrt{n\pi}}; \quad (2.3)$$

$$\binom{n}{k} = \left( \frac{n}{2\pi k(n-k)} \right)^{1/2} \binom{n}{k} \binom{n}{n-k}^{n-k} \times \left[ 1 + \frac{1}{12} \left( \frac{1}{n} - \frac{1}{k} - \frac{1}{n-k} \right) + \frac{1}{288} \left( \frac{1}{n} - \frac{1}{k} - \frac{1}{n-k} \right)^2 + O\left(\frac{1}{kn(n-k)}\right) \right] \quad (2.4)$$

$$\sum_k \binom{r}{m+k} \binom{s}{n-k} = \binom{r+s}{m+n} \Rightarrow (\mathbf{b}) \sum_k \binom{r}{m+k} \binom{s}{n+k} = \binom{r+s}{r-m+n} \quad r \in \mathbb{Z}_0, m, n \in \mathbb{Z}. \quad (\text{Vandermonde's conv.}) \quad (3.1)$$

$$\sum_{k=0}^r \binom{r-k}{m} \binom{s+k}{n} = \binom{r+s+1}{m+n+1} \quad n \geq s \geq 0, r, m \geq 0. \quad \sum_k \binom{l}{m+k} \binom{s+k}{n} (-1)^k = (-1)^{l+m} \binom{s-m}{n-l} \quad l \geq 0. \quad (3.2)$$

$$\sum_{k \leq r} \binom{r-k}{m} \binom{s}{k-t} (-1)^k = (-1)^{r+m} \binom{s-m-1}{r-t-m} \quad r, m \in \mathbb{Z}_{\geq 0}, t \in \mathbb{Z} \xrightarrow{p, h \in \mathbb{R}_+} \sum_k \binom{p+h-k}{m-k} \binom{p}{k} (-1)^k = \binom{h}{m}, \quad (3.3)$$

$$\sum_{k \geq 0} \binom{n+1}{2k+1} \binom{k}{j} = 2^{n-2j} \binom{n-j}{j} \quad n, j \in \mathbb{Z}_{\geq 0}; \quad (\mathbf{b}) \sum_k \binom{n+\alpha}{k} \binom{n+\beta}{n-k} t^k = (t-1)^n P_n^{(\alpha, \beta)}\left(\frac{t+1}{t-1}\right). \quad \text{P-Jacobi polynomial, G\&R 8.96.} \quad (3.4)$$

$$\sum_{k=0}^{\lfloor r/2 \rfloor} \binom{n}{k} \binom{n+r-2k-1}{n-1} (-1)^k = \binom{n}{r}; \quad \sum_{k=0}^{\lfloor r/2 \rfloor} \binom{n+r-1}{k} \binom{2n+2r-2k-2}{2n+r-1} (-1)^k = 2^{r-1} \binom{n+r-1}{n}; \quad n, r \in \mathbb{N}; \quad (3.5)$$

$$\sum_k \binom{n}{k} \binom{r-sk}{n} (-1)^k = s^n \quad r, s \in \mathbb{R}; \quad (\mathbf{b}) \sum_{k=0}^t \frac{\binom{t+v-r-s}{t-k-r}}{\binom{t+v}{t-k}} = \frac{\binom{t+v+1}{r+s+1}}{\binom{r+s}{r}}, \quad s \geq v; t+v \geq r+s; \quad (3.6)$$

$$\sum_k \frac{\binom{m}{k}}{\binom{n}{k}} = \frac{n+1}{n-m+1} \quad n \geq m; \quad (\mathbf{b}) \sum_{k=1}^s \frac{\binom{k+x}{k}}{\binom{k+y}{k}} = \sum_{k=1}^s \frac{(x+1)^{\bar{k}}}{(y+1)^{\bar{k}}} = \frac{(s+1+y)(s+1+x)^{s+1}}{(x+1-y)(s+1+y)^{s+1}} - \frac{x+1}{x-y+1} \quad (3.7)$$

$$\sum_{k=0}^s \frac{\Gamma(x+k)}{\Gamma(y+k)} = \frac{y+s}{x+1-y} \frac{\Gamma(s+1+x)}{\Gamma(s+1+y)} - \frac{y-1}{x+1-y} \frac{\Gamma(x)}{\Gamma(y)}. \quad \text{same as (3.7)(b)} \quad (3.8)$$

$$\sum_{k=0}^s \frac{\binom{a+n}{k}}{\binom{b+n}{k}} = \frac{(b+n-s)(a+n)^{s+1}}{(a-1-b)(b+n)^{s+1}} - \frac{b+n+1}{a-1-b} \quad (3.9)$$

$$\sum_k \binom{r+tk}{k} \binom{s-tk}{n-k} \frac{r}{r+tk} = \binom{r+s}{n}; \quad \sum_k \binom{n}{k} (r+tk)^{k-1} (s-tk)^{n-k} = \frac{(r+s)^n}{r} \quad r, s \in \mathbb{R} \quad (3.10)$$

$$\sum_{k \geq 0} \binom{r+tk}{k} \binom{s-tk}{n-k} = \sum_{k \geq 0} \binom{p-k}{n-k} t^k = \begin{cases} t^{p+1} (t-1)^{n-p-1} & \text{integer } p < n \\ (1-t^{n+1})/(1-t) & p = n \\ \text{no closed form when } p > n \text{ or when } p \notin \mathbb{N} \end{cases} \quad p \equiv r+s \quad (3.11)$$

$$\sum_{k \geq 0} \binom{m}{k} \binom{n-k}{n-k-l} (t-1)^k = \sum_{k \geq 0} \binom{m}{k} \binom{n-m}{n-k-l} t^k \quad l, m, n \in \mathbb{Z}_{\geq 0}. \quad (3.12)$$

$$\sum_k \binom{-1/2}{k} \binom{1/2}{n+1-k} (-1)^k = 2^{-n} \binom{n}{\lfloor n/2 \rfloor}. \quad \sum_k \binom{2n}{k} \binom{2n+2}{k} (-1)^k = \frac{(-1)^n (2n)}{n+1} \binom{2n}{n}. \quad (3.13)$$

$$\sum_{i \geq 0} \binom{-1/2}{i} \frac{1}{2i+1} \binom{-1/2}{k-i-1} \frac{1}{2k-2i-1} = \frac{(-1)^{k+1} 2^{2k-1}}{k^2 \binom{2k}{k}} \quad (\mathbf{b}) \sum_{k=0}^m \binom{n+m-2k}{n-k} \binom{2k}{k} \frac{1}{k+1} = \binom{n+m+1}{m}. \quad (3.14)$$

$$\sum_{j \geq 0} \binom{m}{j} \binom{j/2}{n} = (-1)^{n-1} \frac{m}{n} 2^{m-2n} \binom{2n-m-1}{n-1}, \quad (\mathbf{b}) \sum_{j \geq 0} \binom{m}{j} \binom{j/2}{n} (-1)^j = (-1)^n \frac{m}{n} 2^{m-2n} \binom{2n-m-1}{n-1}, \quad n \geq m \quad (3.15)$$

$$\sum_k \binom{x+1}{k} \binom{x+a-1}{k+a-1} \binom{x-k}{n-k} = \binom{x+a-1}{n+a-1} \binom{x+n+a}{n} \quad (\mathbf{b}) \sum_k \binom{x+y+k}{k} \binom{y}{a-k} \binom{x}{b-k} = \binom{x+a}{b} \binom{y+b}{a} \quad (3.16)$$

$$\sum_{k=0}^n (a+rk)q^k = \frac{a-(a+nr)q^{n+1}}{1-q} + \frac{rq(1-q^n)}{(1-q)^2}; \quad \sum_{k=0}^n kq^k = \frac{q}{(1-q)^2} [1-(n+1)q^n+nq^{n+1}] \quad (4.1)$$

$$\sum_{k=0}^n k^2 q^k = \frac{q}{(1-q)^3} [1+q-(n+1)^2 q^n + (2n^2+2n-1)q^{n+1} - n^2 q^{n+2}]; \quad S_r^{(n)}(q) \equiv \sum_{k=1}^n k^r q^k : S_{r+1}^{(n)}(q) = q D_q S_r^{(n)}(q). \quad (4.2)$$

$$\sum_{k=0}^n k(-1)^k = (-1)^n \left[ \frac{n}{2} \right]; \quad \sum_{k=0}^n k^2 (-1)^k = \frac{(-1)^n}{2} n(n+1); \quad \sum_{k=a}^n k^m = \frac{k^{m+1}}{m+1} \Big|_{k=a}^{n+1}, \quad m \neq -1, a \geq 0; \quad (4.3)$$

$$\sum_{k=0}^n k^2 q^k = \frac{q^2}{(1-q)^3} [2-n(n+1)q^{n-1} + 2(n^2-1)q^n - n(n-1)q^{n+1}]; \quad (\mathbf{b}) T_r(q) = \sum_k k^r q^k = \frac{r! q^r}{(1-q)^{r+1}}. \quad (4.4)$$

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + O(n^{-6}). \quad (\mathbf{b}) \Psi(n+1) + \gamma = H_n; \quad (\mathbf{c}) \Psi(1, n) = \frac{\pi^2}{6} - H_{n-1}^{(2)}. \quad (4.5)$$

$$\sum_{k=1}^n H_k = (n+1)H_n - n = (n+1)(H_{n+1} - 1) \quad (\mathbf{b}) \sum_{k=1}^n \binom{k}{m} H_k = \binom{n+1}{m+1} \left[ H_{n+1} - \frac{1}{m+1} \right], \quad (\mathbf{c}) \sum_{k=1}^n \frac{H_k}{k} = \frac{1}{2} [H_n^2 + H_n^{(2)}] \quad (4.6)$$

$$\sum_{k=1}^n \frac{H_k}{k+1} = \frac{1}{2} [H_{n+1}^2 - H_{n+1}^{(2)}]; \quad (\mathbf{b}) \sum_{k=2}^n \frac{H_k}{k-1} = \frac{1}{2} [H_{n-1}^2 + H_{n-1}^{(2)}] + 1 - \frac{1}{n}; \quad (\mathbf{c}) \sum_{k=1}^n H_k^2 = (n+1)H_n^2 - (2n+1)H_n + 2n. \quad (4.7)$$

$$\sum_{k=1}^{2n} (-1)^k H_k = H_n/2; \quad (\mathbf{b}) \sum_{k=1}^n H_k H_{n+1-k} = (n+2)(H_{n+1}^2 - H_{n+1}^{(2)}) - 2(n+1)H_n + 2n. \quad (4.8)$$

$$\sum_{k=0}^n k^m = \frac{1}{1+m} \sum_{k=0}^m \binom{m+1}{k} (n+1)^{m-k+1} B_k = \frac{B_{m+1}(n+1) - B_{m+1}}{1+m} = \frac{n^{m+1}}{m+1} (1 + O(n^{-1})). \quad (4.9)$$

$B_k, B_k(\cdot)$  are Bernoulli numbers, (6.12). and polynomials.  $\zeta(s, m)$  is the Hurwitz function.

$$\sum_{k=1}^n F_k = F_{n+2} - 1; \quad \sum_{k=1}^n k F_k = (n-2)F_{n+1} + (n-1)F_n + 2; \quad \sum_{k=0}^n (-1)^k k F_k = (-1)^n (n+1)F_{n-1} + (-1)^{n-1} F_{n-2} - 2, \quad n \geq 2. \quad (4.10)$$

$$\sum_{k=0}^n F_k F_{n-k} = \frac{1}{5} (nL_n - F_n) = \frac{1}{5} [2nF_{n-1} + (n-1)F_n]. \quad (\mathbf{b}) \sum_{k=0}^n k F_k F_{n-k} = \frac{n}{10} [2nF_{n-1} + (n-1)F_n]. \quad (4.11)$$

$$\sum_{n \geq 1} \prod_{i=1}^n \frac{q}{b+i} = qe^q \sum_{k \geq 0} \frac{(-q)^k}{k!} \frac{1}{b+k+1}, \quad \sum_{n \geq 1} n \prod_{i=1}^n \frac{q}{b+i} = qe^q \sum_{k \geq 0} \frac{(-q)^k}{k!} \frac{q+k+1}{b+k+1}, \quad b \notin -\mathbb{N} \quad (5.1)$$

$$\sum_{n \geq 1} n^2 \prod_{i=1}^n \frac{q}{b+i} = qe^q \sum_{k \geq 0} \frac{(-q)^k}{k!} \cdot \frac{q+(q+k+1)^2}{b+k+1}, \quad b \notin -\mathbb{N} \quad \text{If } b=0 \text{ the sum is } e^q(1+q)q. \quad (5.2)$$

$$\sum_{\substack{n \\ k \neq l}} \frac{1}{k-l} \left[ (-1)^l \binom{n}{l} + (-1)^k \binom{n}{k} \right] = 0, \quad 0 \leq l \leq n; \quad \sum_{n \geq 0} \prod_{i=1}^n \frac{q}{r+i} = \frac{r!}{q^r} (e^q - e_{r-1}(q)), \quad r \in \mathbb{N}; \quad (5.3)$$

$$\sum_{j \geq 0} \frac{(qz)^j}{(z+1)(z+q) \cdots (z+q^j)} = 1, \quad q, z \in \mathbb{C}, |q| > 1 \text{ (or } |q|=1, |z| < 1); \quad (5.4)$$

$$\sum_{j=1}^n \frac{a_j^k}{\prod_{\substack{i=1 \\ i \neq j}}^n (a_j - a_i)} = \begin{cases} 0 & k < n-1 \\ 1 & k = n-1 \\ h_{k-n+1}(\mathbf{a}) & k \geq n \end{cases} \quad \left| \begin{array}{l} \text{distinct } a_i \in \mathbb{C}; \\ h_r(\mathbf{a}) = \sum_{\substack{i_j \geq 0 \\ \sum i_j = r}} \prod_{j=1}^n a_j^{i_j} \text{ symm. poly} \end{array} \right. \implies \sum_{j=1}^n \prod_{\substack{k \neq j \\ k=1}}^n \frac{z-a_j}{a_k - a_j} = 1. \quad (5.5)$$

$\sigma_k(t_1, \dots, t_n) = [z^{n-k}](z+t_1) \cdots (z+t_n)$ , the  $k$ th elementary symmetric polynomial in the  $n$  variables  $t_j, 1 \leq k \leq n$ .

$$\text{Let } S_m \equiv \sum_{i=1}^n t_i^m, \text{ then } S_m = \sum_{i_1+2i_2+\dots+ni_n=m} (-1)^{i_2+i_4+i_6+\dots} \frac{(i_1+i_2+\dots+i_n-1)! m}{i_1! i_2! \cdots i_n!} \sigma_1^{i_1} \sigma_2^{i_2} \cdots \sigma_n^{i_n}, \text{ Waring formula} \quad (5.6)$$

$$\sum_{r \geq 0} \frac{(uv)^r z^{\binom{r}{2}}}{\prod_{i=0}^r (1-vz^i)} = \sum_{i \geq 0} v^i \prod_{j=0}^{i-1} (1+uz^j); \quad \text{viz. (5.8)(a).} \quad \text{The case } k=n-1 \text{ in (5.5) is Good's identity.} \quad (5.7)$$

$$\sum_{n \geq 0} \frac{u^n}{\prod_{i=1}^n (1-q^i)} = \frac{1}{\prod_{i \geq 0} (1-uz^i)}, \quad \sum_{r \geq 0} \frac{u^r z^{\binom{r}{2}}}{\prod_{i=1}^r (1-z^i)} \stackrel{(a)}{=} 1 + u \sum_{i \geq 0} z^i \prod_{j=0}^{i-1} (1+uz^j) = \prod_{k \geq 0} (1+uz^k), \text{ Euler identities} \quad (5.8)$$

$$\frac{(uz)_\infty}{(z)_\infty} = \sum_{n \geq 0} \frac{(u)_n z^n}{(q)_n}, \quad \text{with } (a)_n = (a; q)_n \equiv \prod_{i=0}^{n-1} (1 - aq^i); \quad (a; q)_0 = 1; \quad (a; q)_t = \frac{(a; q)_\infty}{(aq^t; q)_\infty}, \quad \text{Cauchy identity} \quad (5.9)$$

$$\sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n (q)_n} t^n = \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{k \geq 0} \frac{(c/b)_k (t)_k}{(at)_k (q)_k} b^k \quad \text{Heine's transformation} \quad (5.10)$$

$$\sum_{k=1}^n a_k \Delta b_k = a_k b_k \Big|_1^{n+1} - \sum_{k=1}^n (\Delta a_k) b_{k+1} \quad \text{summation by parts.} \quad (6.1)$$

$$\sum_{1 \leq i, j \leq n} \min(x_i, x_j) = 2 \left( \sum_{i=1}^n i x_{(i)} - \sum_{i=1}^n x_i \right), \quad \text{where } x_{(1)} \geq x_{(2)} \geq \dots \geq x_{(n)}. \quad (6.2)$$

$$\sum_{m=1}^M \sum_{n=1}^N \min(m, n) = \frac{1}{6} N(N+1)(3M-N+1). \quad \sum_{m=1}^M \sum_{n=1}^N \max(m, n) = \frac{1}{6} N(N^2-1) + \frac{1}{2} MN(M+1). \quad M \geq N \quad (6.3)$$

$$\int e^{-\lambda t} t^n dt = -\frac{e^{-\lambda t}}{\lambda^{n+1}} n! \sum_{i=0}^n \frac{(t\lambda)^i}{i!}; \quad \int_{t=0}^{\infty} e^{-\lambda t} (a+t)^j dt = \frac{j!}{\lambda^{j+1}} \sum_{i=0}^j \frac{(a\lambda)^i}{i!}; \quad (6.4)$$

$$\int_{t=0}^1 t^{z-1} (1-t)^{w-1} dt = \int_{u=0}^{\infty} \frac{u^{z-1}}{(1+u)^{w+z}} du = B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(w+z)}, \quad (6.5)$$

$$\int_{-\infty}^{\infty} x^{2j} e^{-ax^2} dx = \frac{1}{a^{(2j+1)/2}} \Gamma\left(j + \frac{1}{2}\right) = \frac{\sqrt{\pi}(2j)!}{\sqrt{a}(4a)^j j!} \quad (\mathbf{b}) \quad \mathbb{E}\left[(\mathbb{N}(0, \sigma^2))^{2j}\right] = \sigma^{2j} \frac{(2j)!}{2^j j!} = \sigma^{2j} (2j-1)!! \quad (6.6)$$

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi/a}, \quad (\mathbf{b}) \quad \int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\pi/a} e^{b^2/4a}, \quad (\mathbf{c}) \quad \int_0^{\infty} e^{-ax^2+bx} dx = \sqrt{\pi/a} e^{b^2/4a} \Phi(b/\sqrt{2a}). \quad (6.7)$$

$$\int_0^{\infty} x e^{-ax^2+bx} dx = \frac{1}{2a} + \frac{b}{a} \sqrt{\pi/a} e^{b^2/4a} \Phi(b/\sqrt{2a}), \quad (\mathbf{b}) \quad \int_{-\infty}^{\infty} x e^{-ax^2+bx} dx = \frac{b}{a} \sqrt{\pi/a} e^{b^2/4a} \quad (6.8)$$

$$\int_0^{\infty} x^{2k} e^{-ax^2} dx = \frac{(2k)! \sqrt{\pi/a}}{2(4a)^k k!}, \quad (\mathbf{b}) \quad \int_0^{\infty} x^{2k+1} e^{-ax^2} dx = \frac{k!}{2a^{k+1}} \quad \text{see (6.6)} \quad (6.9)$$

$$\sum_{k=1}^n f(k) = \int_1^n f(x) + 1/2[f(n) + f(1)] + \sum_{j=1}^m \frac{B_{2j}}{(2j)!} [f^{[2j-1]}(n) - f^{[2j-1]}(1)] + \frac{1}{(2m+1)!} \int_1^n B_{2m+1}(\{x\}) f^{[2m+1]}(x) dx. \quad (6.10)$$

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = n: \quad \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil: \quad \left\lceil \frac{n-1}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor - 1 \quad (6.11)$$

$$B_{k \geq 0}: 1, -1/2, 1/6, (\text{even}) -1/30, 1/42, -1/30, 5/66, -691/2730, 7/6, -3617/510, 43867/798, -174611/330 \quad (6.12)$$

$$\zeta(1-s) = \frac{2}{(2\pi)^s} \cos\left(\frac{s\pi}{2}\right) \Gamma(s) \zeta(s); \quad \zeta(1-n) = (-1)^{n+1} B_n/n, \quad n \geq 1. \quad (6.13)$$

$$\sum_{k=1}^m k^{-s} = \zeta(s) - \zeta(s, m+1); \quad v \zeta(v+1, m+1) \sim \sum_{j \geq 0} \frac{v^j B_j}{j! m^{j+v}} = \frac{1}{m^v} - \frac{v}{2m^{v+1}} + \frac{v(v+1)}{12m^{v+2}} + \dots; \quad m \rightarrow \infty \quad (\text{see (6.12)}) \quad (6.14)$$

$$\ln(1+x) = -\sum_{r \geq 1} \frac{(-x)^r}{r} = \sum_{r \geq 1} \frac{1}{r} x^r (1-x^r), \quad |x| < 1. \quad \ln(x) \stackrel{(x \geq 1/2)}{=} \sum_{k \geq 1} \frac{(x-1)^k}{k x^k} \stackrel{(x > 0)}{=} \sum_{k \geq 0} \frac{1}{2k+1} \left(\frac{x^2-1}{x^2+1}\right)^{2k+1}. \quad (6.15)$$

$$1 - \Phi(x) \sim \frac{\phi(x)}{x} (1 - x^{-2} + 3x^{-4} \dots) \quad (\text{A\&S, 26.2.12}), \quad \text{erf}(x) = 2\Phi(x\sqrt{2}) - 1. \quad (6.16)$$

## INEQUALITIES

$$\mathbf{Markov:} \quad \frac{\mathbb{E}|X|^r - a^r}{a.s. \sup |X|^r} \leq \mathbb{P}\{|X| \geq a\} \leq \mathbb{E} \left[ \left( \frac{|X|}{a} \right)^n \right] \quad r, n \geq 0. \quad (7.1)$$

This is the best possible when the only information is that  $X$  is non-negative and with finite mean. When it is also known that  $X$  is unimodal about  $M$ , then  $\mathbb{P}[|X - M| \geq a] < \left(\frac{n}{n+1}\right)^n \mathbb{E} \left[ \left| \frac{X - M}{a} \right|^n \right]$ . □

$$\text{Inequality (7.1) for integer rv's and } n = 1 \text{ becomes: } \mathbb{P}(X > 0) \leq \mathbb{E}[X]. \text{ also for any rv } \mathbb{P}(X = 0) \leq \sigma^2/\mathbb{E}[X^2]. \quad (7.2)$$

$$\mathbf{Bienaym\`e-Chebyshev:} \quad \mathbb{P}\{|X| \geq t\} \leq t^{-2} \mathbb{E}[X^2] \quad \text{same as the RHS of (7.1) for } n = 2. \quad (7.3)$$

$$\text{Chebyshev –Cantelli: } \mathbb{P}\{X - \mu \geq t\} \leq \frac{\sigma^2}{\sigma^2 + t^2} \quad (7.4)$$

Both (7.1) and (7.3) are special cases of the following, more general inequality (Loève, “Probability Theory” Vol I, p. 159): Let  $X$  be an arbitrary r.v. and  $g$  on  $R$  be a nonnegative Borel Function. If  $g$  is even and nondecreasing on  $[0, +\infty)$  then, for every  $a \geq 0$

$$\text{Loève: } \frac{\mathbb{E}[g(X)] - g(a)}{a.s. \sup g(X)} \leq \mathbb{P}[|X| \geq a] \leq \frac{\mathbb{E}[g(X)]}{g(a)} \quad (7.5)$$

If  $g$  is nondecreasing on  $R$ , then the middle term is replaced by  $\mathbb{P}[X \geq a]$ , where  $a$  is an arbitrary number.  $\square$

Let  $X_1, X_2, \dots, X_n$  be independent rv's with  $\mathbb{E}X_i^2 < \infty$ , and  $S_k$  the sum of the first  $k$   $X_i$ 's. Then, for any  $\varepsilon > 0$

$$\text{Kolmogorov: } \mathbb{P}\left(\max_{1 \leq k \leq n} |S_k - \mathbb{E}[S_k]| \geq \varepsilon\right) \leq \frac{\sigma_n^2}{\varepsilon^2} \quad \sigma_n^2 \equiv V[S_n]. \quad (7.6)$$

This also holds when  $S_n$  is a martingale. For  $n = 1$  (7.6) reduces (7.3).  $\square$

Let  $X_i, 1 \leq i \leq n$  be i.i.d. bounded rv's,  $0 \leq X_i \leq 1$ ,  $\mu = \mathbb{E}[S = \sum X_i]/n$ , then, for  $0 < t < 1 - \mu$ :

$$\text{Hoeffding: } \mathbb{P}[S - n\mu \geq nt] \leq \left[ \left( \frac{\mu}{\mu + t} \right)^{\mu + t} \left( \frac{1 - \mu}{1 - \mu - t} \right)^{1 - \mu - t} \right]^n \leq \exp(-nt^2 g(\mu)) \leq \exp(-2nt^2), \quad (7.7)$$

where

$$g(\mu) = \begin{cases} \frac{1}{1-2\mu} \log\left(\frac{1-\mu}{\mu}\right) & 0 < \mu < \frac{1}{2} \\ \frac{1}{2\mu(1-\mu)} & \frac{1}{2} \leq \mu < 1 \end{cases} \quad \text{Note: } g(\mu) \geq 2, \quad \forall \mu.$$

For  $t > 1 - \mu$  the first bound in the right-hand-side in (7.7) is 0, and for  $t = 1 - \mu$  it equals  $\mu^n$ . (Hoeffding, *J. Am. Stat. Ass.* **58**, 13–30, 1963). The right-most inequality is often named after **Chernoff**.  $\square$

The combinatorial **Bonferroni** inequalities are easily adapted probability measures (Comtet, pp. 192–3). Let  $A = \{A_i, 1 \leq i \leq n\}$  be a set of events in some probability space  $\Omega$ , associated with a measure  $\mathbb{P}$ , and define the probabilities  $S_k$  as the sum of probabilities of occurrence of the intersections of all  $k$ -strong subsets of these events:  $S_0 \equiv \mathbb{P}(\Omega)$ ,  $S_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \mathbb{P}(A_{i_1} A_{i_2} \dots A_{i_k})$ . Then the sum

$\sum_{k=1}^n (-1)^{k-1} S_k$  satisfies the *alternating inequalities*:

$$\text{Bonferroni: } (-1)^k \left[ \mathbb{P}\left(\bigcup_{i=1}^n A_i\right) + \sum_{j=1}^k (-1)^j S_j \right] \geq 0 \quad (-1)^{k+1} \left[ \mathbb{P}\left(\bigcap_{i=1}^n \bar{A}_i\right) + \sum_{j=1}^k (-1)^{j+1} S_j \right] \geq 0 \quad 1 \leq k \leq n \quad (7.8)$$

For  $k = 1$  the above reduces **Boole**'s inequality.  $\square$

Let  $\mu_i$  be moments of  $X$ , and  $L(s)$  its LST.  $L(s)$  is absolutely monotone, and

$$\text{Laplace-Stieltjes Transform: } \sum_{i=0}^{2n-1} \frac{\mu_i(-s)^i}{i!} \leq L(s) \leq \sum_{i=0}^{2n} \frac{\mu_i(-s)^i}{i!} \quad \forall n \geq 1. \quad (7.9)$$

$$\text{Jensen: } \mathbb{E}[u(X)] \geq u(\mathbb{E}[X]) \quad \text{when } u(\cdot) \text{ is a convex function.} \quad \square \quad (7.10)$$

$$\text{Chebyshev: } n \sum_{k=1}^n a_k b_k \geq \left( \sum_{k=1}^n a_k \right) \left( \sum_{k=1}^n b_k \right) \quad \text{where } \{a_i\} \text{ and } \{b_i\} \text{ are monotonic, both increasing or decreasing,} \quad (7.11)$$

If the senses of monotonicity are opposite, reverse the inequality sign.  $\square$

$$\text{Hölder: } \sum_{k=1}^n |a_k b_k| \leq \left( \sum_{k=1}^n |a_k|^p \right)^{1/p} \left( \sum_{k=1}^n |b_k|^q \right)^{1/q} \quad \text{for } p > 1, q > 1, \text{ and } p^{-1} + q^{-1} = 1. \quad (7.12)$$

The **Cauchy-Bunyakovsky-Schwartz (CBS)** inequality is obtained for  $p = q = 2$ . The analogue holds for integrals.  $\square$

The CBS inequality also leads the following (in Vitter & Krishnan, *JACM* **43**, #5 p.785):

$$\text{Amit-Miller: } \left( \sum_{k=1}^n |p_k - r_k| \right)^2 \leq 2 \sum_{k=1}^n p_k \ln(p_k/r_k), \quad \text{for any two } n \text{ term PMFs } \mathbf{p} \text{ and } \mathbf{r} \quad (7.13)$$

$$\text{Gibbs: } \sum_{k=1}^n p_k \log(p_k/r_k) \geq 0, \quad \text{for any two } n \text{ term PMFs } \mathbf{p} \text{ and } \mathbf{r} \text{ (with equality if the PMFs coincide).} \quad (7.14)$$

$$\text{Minkowski-I: } \left( \sum_{k=1}^n (a_k + b_k)^p \right)^{1/p} \leq \left( \sum_{k=1}^n a_k^p \right)^{1/p} + \left( \sum_{k=1}^n b_k^p \right)^{1/p} \quad \text{for } p > 1 \text{ and } a_k, b_k > 0, \forall k. \quad (7.15)$$

The analogue holds for integrals. □

**Minkowski-II:** 
$$\prod_{k=1}^n (1+a_k) \geq \left(1 + \left(\prod_{k=1}^n a_k\right)^{1/n}\right)^n \quad \text{for } a_k \geq 0 \text{ for all } k \text{ and integer } n. \quad (7.16)$$

**Carleman:** 
$$\sum_{k=1}^n (a_1 a_2 \cdots a_k)^{1/k} \leq e \sum_{k=1}^n a_k \quad \text{for } a_k \geq 0 \text{ for all } k, \text{ and an integer } n. \quad (7.17)$$

On the interval  $[a, b]$  let  $f(\cdot)$  be real and continuous and let there be a family (not necessarily finite) of real and continuous orthonormal functions,  $\{\psi_r(\cdot)\}$ , then

**Bessel:** 
$$\sum_r \left[ \int_a^b f(x) \psi_r(x) dx \right]^2 \leq \int_a^b f^2(x) dx, \quad \text{The conditions on } f, \text{ which may be complex, can be relaxed.} \quad (7.18)$$

**Hilbert:** 
$$\sum_{m,n} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} A^{1/p} B^{1/q}, \quad \text{when } p, q > 1, 1/p + 1/q = 1, \Sigma_m a_m^p \leq A, \Sigma_n b_n^q \leq B. \quad (7.19)$$

unless one of the series vanishes identically. (More on this in Hardy, Littlewood and Pólya, Chapter IX). □

**interpolation:** 
$$\frac{n}{n+1} \frac{m}{m-1} \left(\frac{n-1}{n}\right)^{n-m+1} \leq 1, \quad 1 < m < n. \quad f(x) = x^p(1-x)^{1-p}, \quad p \in (0, 1) \text{ is maximized at } x = p. \quad (7.20)$$

**The wedge inequality:** 
$$|(a \vee b) - (c \vee d)| \leq |a - c| \vee |b - d| \quad a, b, c, d \in \mathbb{R}; \quad \vee : \text{selection of the maximum :} \quad (7.21)$$

The inequality holds also for functions instead of numbers, with  $|\cdot|$  replaced by any  $p$ -norm,  $p > 0$ . □

**The ratio inequality:** 
$$\sum_{i=1}^n a_i / \sum_{j=1}^n b_j \leq \max_k \frac{a_k}{b_k} \quad a_k, b_k \in \mathbb{R}^+, \quad (7.22)$$

Let  $A, G$  and  $H$  denote the arithmetic, geometric and harmonic means of  $\{a\} = \{a_i, 1 \leq i \leq n\}$ , given by

**The Three Means:** 
$$A = \sum a_i/n, \quad G = [\prod a_i]^{1/n}, \quad H = [\sum a_i^{-1}/n]^{-1}. \quad (7.23)$$

The standard relation  $A \geq G \geq H$  can be generalized as follows: Let  $M_t\{a\} \equiv (\frac{1}{n} \sum a_i^t)^{1/t}$ , then

**Power-means (Arnon):** 
$$t_1 > t_2 \implies M_{t_1}\{a\} \geq M_{t_2}\{a\}. \quad \text{Known as Lyapunov inequality in probability.} \quad (7.24)$$

$A, G, H$  correspond to  $t = 1, 0, -1$ . As  $t \rightarrow \pm\infty$ ,  $M_t\{a\}$  approaches  $\max\{a\}$  and  $\min\{a\}$  resp. The power-means ineq. can be further generalized with weights: Let  $\mathbf{p}$  a PMF, and  $\mathbb{E}_t^{\mathbf{p}}\{a\} \equiv (\sum p_i a_i^t)^{1/t}$ , then

**Weighted power-means:** 
$$t_1 > t_2 \implies \mathbb{E}_{t_1}^{\mathbf{p}}\{a\} \geq \mathbb{E}_{t_2}^{\mathbf{p}}\{a\}. \quad \square \quad (7.25)$$

Let  $N_t\{a\} \equiv (\sum a_i^t)^{1/t}$ , with all  $a_i > 0$ , then

**Norm monotonicity:** 
$$t_1 > t_2 \implies N_{t_1}\{a\} \leq N_{t_2}\{a\}. \quad (7.26)$$

**The Jackknife inequality** (Efron & Stein *Ann. Statist.* **9**, 586–596, 1981).

Let  $\{X_i\}$ ,  $1 \leq i \leq n$  be i.i.d and  $H$  a symmetric measurable function over  $n-1$ -dimensional vectors.

Define  $S_i = H(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$ , and  $S = \sum_{i=1}^n S_i/n$ . Then

**The Jackknife inequality:** 
$$V[H(X_1, \dots, X_{n-1})] \leq \mathbb{E} \left[ \sum_{i=1}^n (S_i - S)^2 \right]. \quad (7.27)$$

**Shachnai:** 
$$0 < q < 1, a_i \geq 0, \quad \sum_{i=1}^n a_i q^i \sum_{j=1}^{i-1} a_j \leq \frac{1+3q}{2(1-q)} \sum_{i=1}^n a_i^2 q^i \quad (7.28)$$

Shown in A. Bar-Noy, M. Bellare, M. Halldorsson, H. Shachnai, T. Tamir, *Inf. and Comput.*, **140**, 1998, 183–202.

$$\frac{4^n}{2\sqrt{n}} \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt{3n+1}}; \quad \text{(b)} \quad \left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left\{ \frac{n^n}{k^k (n-k)^{n-k}}, \quad \left(\frac{en}{k}\right)^k \right\}. \quad (7.29)$$

**Logarithm:** 
$$\lim_{x \rightarrow \infty} x^{-a} \log x = 0, \quad \text{(b)} \quad \lim_{x \rightarrow 0} x^a \log x = 0, \quad \Re a > 0 \quad \text{(c)} \quad \ln x \leq n(x^{1/n} - 1), \quad x > 0, n \in \mathbb{Z}_+ \quad (7.30)$$

$$\frac{x}{x+1} < \ln(1+x) < x, \quad 0 \neq x > -1, \quad \text{(b)} \quad x < -\ln(1-x) < \frac{x}{1-x}, \quad 0 \neq x < 1, \quad \text{(c)} \quad \ln x \leq x-1, \quad x > 0 \quad (7.31)$$

$$|\log(1+z)| \leq -\log(1-|z|). \quad |z| < 1. \quad \text{(b)} \quad \left(1 + \frac{1}{x}\right)^x \quad x > 0 \text{ is monotonic incr.} \quad (7.32)$$

**Exponent:** 
$$e^{-\frac{x}{1-x}} < 1-x < e^{-x}, \quad x \in [0, 1), \quad \text{(b)} \quad \frac{1}{1-x} > (x < 1) e^x > 1+x, \quad \text{(c)} \quad \frac{x}{1+x} < 1 - e^{-x} < x, \quad (x > -1). \quad (7.33)$$

$$1+x > e^{\frac{x}{1+x}} \quad x > -1, \quad \text{(b)} \quad e^x > \left(1 + \frac{x}{y}\right)^y > e^{\frac{xy}{x+y}} \quad x, y > 0 \quad \text{(c)} \quad |e^z - 1| \leq e^{|z|} - 1 \leq |z|e^{|z|} \quad \text{all } z. \quad (7.34)$$

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Let  $m_a = \mathbb{E}[|X - \mathbb{E}X|^a]$ ,  $a \in \mathbb{R}$ , then  $m_a^{1/a}$  is a monotonic increasing function in  $a$ . See (7.24) (8.1)

$$\int_{p=0}^1 F_p^r dp = \mathbb{E}(X^r), \text{ where } F_p \text{ is the } p\text{-fractile of the rv } X : \mathbb{P}(X \leq F_p) = p. \quad (8.2)$$

$$\sum_{i \geq 1} \int_{t=0}^u [1 - G(u-t)] g^{*i}(t) dt = G(u), \quad g(t) = \frac{dG(t)}{dt}; \quad 1 - \Phi(x) \sim \frac{\phi(x)}{x} (1 - x^{-2} + 3x^{-4} \dots) \quad (\text{A \& S, 26.2.12}). \quad (8.3)$$

For the  $M/G/1$  queue: PGF of customers in system, mean value and variance, where  $S(t)$  is the LST of the service time distribution and  $\sigma_r$  is  $\mathbb{E}(S^r)$ :

$$g(z) = p_0 \frac{\tilde{S}(z)(z-1)}{z - \tilde{S}(z)}, \quad \tilde{S}(z) \equiv S(\lambda - \lambda z), \quad p_0 = 1 - \lambda \mathbb{E}(S) = 1 - \rho, \quad \mathbb{E}(X) = \rho + \frac{\lambda^2 \sigma_2}{2(1-\rho)}, \quad \mathbb{E}(W) = \frac{\lambda \sigma_2}{2(1-\rho)} \quad (8.4)$$

$$\mathbb{E}(X^2) = \mathbb{E}(X) + \frac{\lambda^3 \sigma_3 + 3\lambda^2 \sigma_2}{3(1-\rho)} + \frac{\lambda^4 \sigma_2^2}{2(1-\rho)^2}, \quad W(s) = \frac{s(1-\rho)}{\lambda S(s) - \lambda + s}.$$

For the  $M/G/\infty$  queue: if arrivals are in batches where the batch size has the PGF  $A(z)$ , and  $F(\cdot)$  is the service time CDF, the number in queue has the Poisson distribution only when  $A(z) = z$ :

The Expected length of a bp and of the number of customers served in a bp are given by

$$\mathbb{E}[B] = \frac{e^{\lambda \xi} - 1}{\lambda}, \quad \mathbb{E}[N] = \alpha e^{\lambda \xi}, \quad \alpha = A'(1), \quad \xi \equiv \int_{t=0}^{\infty} (1 - A(F(t))) dt. \quad (8.5)$$

When  $A(z) = z$ ,  $\xi = \mathbb{E}[S]$ ,  $\mathbb{E}[B] = (e^\rho - 1)/\lambda$ ,  $\mathbb{E}[N] = e^\rho$ . (Shanbhag, J. Appl. Prob. **3**, 274–279 (1966)).

$$\text{Volume of an } n\text{-dimensional ball of radius } R: \quad V_n(R) = 2^{\lceil n/2 \rceil} \pi^{\lfloor n/2 \rfloor} R^n / n! = \pi^{n/2} R^n / \Gamma(n/2 + 1). \quad A_n(R) = dV_n(R)/dR \quad (8.6)$$

$$\text{Number of grid points in corner of } d\text{-dimensional cube of side } m: \quad \binom{m+d}{m} \quad (8.7)$$

$$\text{Moments of } N(a, \sigma^2): \quad \mathbb{E}(X - a)^{2r} = (2r)! \sigma^{2r} / 2^r \Gamma(r + 1). \quad \mathbb{E}|X - a|^r = 2^{r/2} \Gamma((r+1)/2) \sigma^r / \sqrt{\pi}, \text{ odd } r \quad (8.8)$$

The eigenvalues of an  $n$ -circulant with first column  $\underline{c}$  are given by

$$\underline{\lambda} = \sqrt{n} \mathbf{W} \underline{c}, \text{ where } W_{jk} = \exp\left(i \frac{2\pi}{n} jk\right). \quad (8.9)$$

## GENERIC EQUATIONS

The general linear differential equation of first order:

$$y'(x) + h(x)y(x) = g(x), \quad (9.1e)$$

with  $h(\cdot)$  and  $g(\cdot)$  known functions, and a prescribed boundary value  $y_0 = y(x_0)$ , has the general solution

$$y(x) = e^{-\int h dx} \left( C + \int g e^{\int h dx} dx \right), \quad \square \quad (9.1s)$$

where the constant  $C$  is determined from the boundary value.

The Bernoulli equation is  $y' + p(x)y = q(x)y^n$ ; the substitution  $v = y^{1-n}$  linearizes it to  $v' = (n-1)[p(x)v - q(x)]$ .

Homogeneous Riccati equation with constant coefficients, aka logistic equation (a particular case of Bernoulli equation):

$$f'(x) = af^2(x) + bf(x), \quad f(0) = \alpha, \text{ where } a \text{ and } b \text{ are constants, has the solution} \quad (9.2e)$$

$$f(x) = \frac{\alpha e^{bx}}{1 - a\alpha b^{-1}(e^{bx} - 1)}, \quad (\text{Goulden and Jackson, 1984}). \quad (9.2s)$$

The general linear first order difference equation

$$a_{t+1} = \alpha_t + \beta_t a_t; \quad t \geq m, \text{ with } a_m, \{\alpha_k\}, \{\beta_k\} \text{ known,} \quad (9.3e)$$

has the solution

$$a_t = \sum_{j=m}^{t-1} \alpha_j \prod_{i=j+1}^{t-1} \beta_i + a_m \prod_{i=m}^{t-1} \beta_i \quad t \geq m \quad (9.3s)$$

Special cases: Constant  $\alpha$  and  $\beta$

$$a_{t+1} = \alpha + \beta a_t; \quad t \geq m \implies a_t = \frac{\alpha}{1-\beta} + \beta^{t-m} \left( a_m - \frac{\alpha}{1-\beta} \right), \quad t \geq m. \quad (9.3c)$$

Pure linear  $\beta_t$ , constant  $\alpha_t$ :

$$a_{i+1} = b + i c a_i, \quad i \geq 1, \quad a_1 \text{ given} \quad (9.5e)$$

$$a_i = (i-1)! c^{i-1} [a_1 + b(e_{i-1}(1/c) - 1)], \quad i \geq 1. \quad e_r(u) - \text{the incomplete exponential function.} \quad (9.5s)$$

Functional equation:  $a(x)$  and  $b(x)$  given functions and  $|p| \neq 1$ .

$$f(x) = a(x) + b(x)f(px), \quad A = \lim_{i \rightarrow \infty} p^i, \quad f(A) \text{ known.} \quad (9.6e)$$

In the region where the following sums and products converge, the solution is given by

$$f(x) = \sum_{j \geq 0} a(p^j x) \prod_{i=0}^{j-1} b(p^i x) + f(A) \prod_{i \geq 0} b(p^i x) \quad (9.6s)$$

$$T(n) = n(\lg n)^{q-1} + 2T(n/2), \quad q \in \mathbb{N} \implies T(n) \sim (n/q)(\lg n)^q. \quad (9.7)$$

$$\text{Standard quadratic equation: } ax^2 + bx + c = 0 \implies x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (9.8)$$