$$\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k}; \quad \textbf{\textit{(b)}} \binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1} = \frac{r-k+1}{k}\binom{r}{k-1} = \frac{r}{r-k}\binom{r-1}{k}. \tag{1.1}$$

$$\binom{r}{k} = \binom{r}{r-k}; \qquad \textbf{\textit{(b)}} \binom{r}{k, m-k, r-m} = \binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k} = \binom{r}{m-k}\binom{r-m+k}{k},\ 0 \le k \le r \in \mathbb{Z}. \tag{1.2}$$

$$\binom{-r}{k}(-1)^k = \binom{r+k-1}{k}; \qquad \textbf{\textit{(b)}} \binom{k+1/2}{n} = \begin{cases} 4^{-n}\binom{2n}{n}(-1)^{n+k-1}\binom{n}{k+1}\binom{2n}{2k+2}^{-1} & k < n \\ 4^{-n}\binom{2n}{n}\binom{2k+1}{2n}\binom{k}{n}^{-1} & k \ge n \end{cases} \tag{1.3}$$

$$\sum_{k=0}^{n}\binom{r+k}{r} = \binom{r+n+1}{n}; \qquad \textbf{\textit{(b)}} \sum_{k=m}^{n}\binom{r}{k}(-1)^k = \binom{r-1}{m-1}(-1)^m + \binom{r-1}{n}(-1)^n; \tag{1.4}$$

$$\sum_{k=0}^{m}\binom{k+n}{t} = \binom{m+n+1}{t+1} - \binom{n}{t+1} \quad \Rightarrow \quad \textbf{\textit{(b)}} \sum_{k=1}^{n}\binom{a-k}{t} = \binom{a}{t+1} - \binom{a-n}{t+1}, \quad a-t \ge n. \tag{1.5}$$

$$\sum_{k=m}^{n}\binom{k}{m} = \binom{n+1}{m+1}; \qquad \textbf{\textit{(b)}} \sum_{k=0}^{m} k\binom{k+n}{k} = (n+1)\binom{m+n+1}{m-1}; \qquad \textbf{\textit{(c)}} \sum_{k=0}^{n}\binom{n-k}{k} = \sum_{k\ge 0}\binom{k}{n-k} = F_{n+1}. \tag{1.6}$$

$$\sum_{k=1}^{n} k!\cdot k = (n+1)! - 1; \quad \textbf{\textit{(b)}} \sum_{k=1}^{n} 2k(2k-1)!! = (2n+1)!! - 1; \quad \textbf{\textit{(c)}} \sum_{k=0}^{n}(2k+1)(2k)!! = (2n+2)!! - 2. \tag{1.7}$$

$$\sum_{k=a}^{n} k^{\underline{m}} = \frac{k^{\underline{m+1}}}{m+1}\Big|_{k=a}^{n+1}, \quad m \ne -1,\ a \ge m; \qquad \textbf{\textit{(b)}} \sum_{k}\binom{n}{k}^2 = \binom{2n}{n}; \qquad \textbf{\textit{(c)}} \sum_{k} k\binom{n}{k}^2 = (2n-1)\binom{2n-2}{n-1}; \tag{1.8}$$

$$\sum_{k=0}^{n}\binom{x+k}{k}^{-1} = \frac{x}{x-1}\left[1 - \binom{x+n}{n+1}^{-1}\right]; \qquad \textbf{\textit{(b)}} \sum_{k=0}^{n}(-1)^k\binom{x}{k}^{-1} = \frac{x+1}{x+2} + \frac{x-n}{x+2}\frac{(-1)^n}{\binom{x}{n+1}}. \quad \textbf{\textit{(c)}} \sum_{k=0}^{n}\frac{4^k}{\binom{2k}{k}} = \frac{1}{3} + \frac{2n+1}{3}\frac{4^{n+1}}{\binom{2(n+1)}{n+1}}. \tag{1.9}$$

$$\sum_{k=0}^{n}\binom{k+u}{k} z^k = \frac{1}{u!}\frac{d^u}{dz^u} z^u \frac{1-z^{n+1}}{1-z} = \frac{1 - z^{n+1}\sum_{i=0}^{u}\binom{i+n}{i}(i-z)^i}{(1-z)^{u+1}},\ u \in \mathbb{N}; \quad \textbf{\textit{(b)}} \sum_{k=0}^{n}\binom{1/2}{k}(-1)^k = \binom{-1/2}{n}(-1)^n. \tag{1.10}$$

$$\sum_{k=0}^{m}\binom{k+n-m-1}{k} z^k = \sum_{k=0}^{m}\binom{n}{k} z^k (1-z)^{m-k},\ \begin{matrix} m \le n \in \mathbb{N}; \\ s = \sqrt{1+4z}. \end{matrix} \qquad \textbf{\textit{(b)}} \sum_{k=0}^{n}\binom{n-k}{k} z^k = \frac{(1+s)^{n+1} - (1-s)^{n+1}}{2^{n+1}s}, \tag{1.11}$$

$$\sum_{k\ge 0}\binom{k+u}{k} z^k = \frac{1}{(1-z)^{u+1}}; \qquad \textbf{\textit{(b)}} \sum_{k\ge 0}\binom{s+k}{u} z^k = \frac{z^{u-s}}{(1-z)^{u+1}} - [u < s]\sum_{k=0}^{s-u-1}\binom{u+k}{u} z^{k-(s-u)} \quad u,\ s \in \mathbb{N}; \quad |z| < 1. \tag{1.12}$$

$$\sum_{k=0}^{n}\binom{x}{k}\frac{k\cdot k!}{x^{k+1}} = 1 - \binom{x}{n+1}\frac{(n+1)!}{x^{n+1}}; \qquad \textbf{\textit{(b)}} \sum_{k\ge 1}\frac{x^k}{k\cdot k!} = E_i(x) - \gamma - \ln x = \frac{e^x}{x}\left(1 + O\left(x^{-1}\right)\right). \tag{1.13}$$

$$\sum_{k\ge 0}\binom{r-tk}{k} z^k = \frac{x^{r+1}}{(t+1)x - t}, \qquad \textbf{\textit{(b)}} \sum_{k\ge 0}\binom{r-tk}{k}\frac{r}{r-tk} z^k = x^r, \text{ where } z = x^{t+1} - x^t, \tag{1.14}$$

and the solution is chosen be regular in $z \in \{|z| < |t^t(t+1)^{-(t+1)}|\}$. This is also the region of convergence.

$$\sum_{k=1}^{n}\frac{1}{k}\binom{r+n-k}{r} = \binom{r+n}{n}\left(H_{n+r} - H_r\right). \qquad \textbf{\textit{(b)}} \sum_{k=0}^{n}\binom{n+k}{k} 2^{-k} = 2^n; \qquad \textbf{\textit{(c)}} \sum_{k=0}^{n}\binom{2k}{k} 4^{-k} = (2n+1)\binom{2k}{k} 4^{-n}; \tag{1.15}$$

$$\sum_{k\ge 1}\binom{n}{k}\frac{(-1)^{k+1}}{k}(x+ky)^r = x^r H_n + r x^{r-1} y,\ 0 \le r \le n. \quad \textbf{\textit{(b)}} \sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{x+j} = \frac{1}{x\binom{n+x}{n}},\ x \notin 0, -1, \ldots, -n. \tag{1.16}$$

$$\sum_{k}\binom{n}{k}(-1)^{n-k} k^n = n! \quad \textbf{\textit{(b)}} \sum_{k}\binom{n}{k}(-1)^{n-k} k^{n+1} = \frac{n}{2}(n+1)! \quad \textbf{\textit{(c)}} \sum_{k}\binom{n}{k}(tk+r)^{k-1}(t(n-k)+s)^{n-k} = \frac{(tn+r+s)^n}{r} \tag{1.17}$$

$$c_{m,n} \equiv \sum_{k}\binom{n}{k}(a+bk)^m(-1)^k = n!(-1)^n\sum_{i}\binom{m}{i}\left\{{i \atop n}\right\} a^{m-i} b^i [= 0 \mid m < n]; \quad \textbf{\textit{(b)}} \sum_{m\ge n} c_{m,n} = \frac{1/(1-a)}{\binom{n-(1-a)/b}{n}} := \frac{\delta(1, a-b)}{(1-a)(n+1)} \tag{1.18}$$

needs $|a+bk|<1,\ k\in[n]$

$$\sum_{k=0}^{r-1}\binom{n+1}{k}(r-k)^n(-1)^k = \sum_{k\ge 0}\binom{n-k}{r-1}\left\{{n \atop k}\right\}(-1)^{k+n+r+1} = A(n,r) = \left\langle {n \atop r-1} \right\rangle \text{ Eulerian \#}s. \tag{1.19}$$

$$\sum_{k\ge 0}\binom{2k}{k} z^k = \frac{1}{s}, \qquad \textbf{\textit{(b)}} \sum_{k\ge 0}\binom{2k}{k} k z^k = \frac{2z}{s^3}, \qquad \textbf{\textit{(c)}} \sum_{k\ge 0}\binom{2k}{k}\frac{z^k}{k+1} = \frac{1-s}{2z}, \qquad \textbf{\textit{(d)}} \sum_{k\ge 0}\binom{2k+1}{k} z^k = \frac{1-s}{2zs}, \quad s = \sqrt{1-4z}. \tag{1.20}$$

$$\sum_{k\ge 0}\binom{2k}{k}\frac{z^{2k+1}}{2k+1}=\frac{\sin^{(-1)}(2z)}{2},\qquad \textbf{\textit{(b)}}\sum_{k\ge 0}\binom{2k}{k}\frac{z^k}{2k-1}=-s,\qquad \textbf{\textit{(c)}}\sum_{k\ge 1}\binom{2k}{k}\frac{z^k}{k}=\log\frac{1-s}{z(1+s)},\quad \text{for } s \text{ see (1.20)} \tag{1.21}$$

$$\sum_{k\ge 0}z^k\sum_{r=0}^{\lfloor k/2\rfloor}\binom{k}{r}x^r=\frac{1/2}{1-z(1+x)}\left(1+\frac{1-2xz}{\sqrt{1-4xz^2}}\right)\ \text{When bisected:} \tag{1.22}$$

$$\sum_{l\ge 0}z^l\sum_{r=0}^{l}\binom{2l+1}{r}x^r=\frac{1/2}{1-z(1+x)^2}\left(1+x+\frac{1-x}{\sqrt{1-4xz}}\right);\quad \textbf{\textit{(b)}}\sum_{l\ge 0}z^l\sum_{r=0}^{l}\binom{2l}{r}x^r=\frac{1/2}{1-z(1+x)^2}\left(1+\frac{1-2x(1+x)z}{\sqrt{1-4xz}}\right) \tag{1.23}$$

$$\Gamma(n+1)=n!=\sqrt{2\pi}n^{n+1/2}e^{-n}\left(1+\frac{1}{12n}+\frac{1}{288n^2}+O\big(n^{-3}\big)\right)\quad n!!=2^{\lceil n/2\rceil}\frac{\Gamma(1+n/2)}{\pi^{n/2-\lfloor n/2\rfloor}}.\quad \Gamma\left(\frac{2k+1}{2}\right)=\frac{\sqrt{\pi}(2k)!}{4^k k!} \tag{2.1}$$

$$\frac{\Gamma(b+n+1)}{\Gamma(b)}=(b+n)^{\underline{n+1}};\quad \frac{\Gamma(-n)}{\Gamma(-m)}=\frac{\Gamma(m+1)}{\Gamma(n+1)}(-1)^{n-m},\quad m,n\in\mathbb{N}.\quad \frac{\Gamma(n+a)}{\Gamma(n+b)}=n^{a-b}\left[1+\frac{(a-b)(a+b-1)}{2n}+O\left(n^{-2}\right)\right] \tag{2.2}$$

$$\binom{1/2}{k}=\frac{1}{1-2k}\binom{-1/2}{k};\quad \binom{n-1/2}{n}=(-1)^n\binom{-1/2}{n}=4^{-n}\binom{2n}{n}\approx\frac{1}{\sqrt{n\pi}}; \tag{2.3}$$

$$\binom{n}{k}=\left(\frac{n}{2\pi k(n-k)}\right)^{1/2}\left(\frac{n}{k}\right)^k\left(\frac{n}{n-k}\right)^{n-k}\times\left[1+\frac{1}{12}\left(\frac{1}{n}-\frac{1}{k}-\frac{1}{n-k}\right)+\frac{1}{288}\left(\frac{1}{n}-\frac{1}{k}-\frac{1}{n-k}\right)^2+O\left(\frac{1}{kn(n-k)}\right)\right] \tag{2.4}$$

$$\sum_k\binom{r}{m+k}\binom{s}{n-k}=\binom{r+s}{m+n}\Rightarrow \textbf{\textit{(b)}}\sum_k\binom{r}{m+k}\binom{s}{n+k}=\binom{r+s}{r-m+n}\quad r\in\mathbb{Z}_0,\ m,n\in\mathbb{Z}.\ \text{(Vandermonde's conv.)} \tag{3.1}$$

$$\sum_{k=0}^{r}\binom{r-k}{m}\binom{s+k}{n}=\binom{r+s+1}{m+n+1}\quad n\ge s\ge 0,\ r,m\ge 0.\quad \sum_k\binom{l}{m+k}\binom{s+k}{n}(-1)^k=(-1)^{l+m}\binom{s-m}{n-l}\quad l\ge 0. \tag{3.2}$$

$$\sum_{k\le r}\binom{r-k}{m}\binom{s}{k-t}(-1)^k=(-1)^{r+m}\binom{s-m-1}{r-t-m}\ r,m\in\mathbb{Z}_{\ge 0},\ t\in\mathbb{Z}\underset{p,h\in\mathbb{R}_+}{\Longrightarrow}\sum_k\binom{p+h-k}{m-k}\binom{p}{k}(-1)^k=\binom{h}{m}, \tag{3.3}$$

$$\sum_{k\ge 0}\binom{n+1}{2k+1}\binom{k}{j}=2^{n-2j}\binom{n-j}{j}\ n,j\in\mathbb{Z}_{\ge 0};\quad \textbf{\textit{(b)}}\sum_k\binom{n+\alpha}{k}\binom{n+\beta}{n-k}t^k=(t-1)^nP_n^{(\alpha,\beta)}\left(\frac{t+1}{t-1}\right).\ \text{P-Jacobi polynomial, G\&R 8.96.} \tag{3.4}$$

$$\sum_{k=0}^{\lfloor r/2\rfloor}\binom{n}{k}\binom{n+r-2k-1}{n-1}(-1)^k=\binom{n}{r};\quad \sum_{k=0}^{\lfloor r/2\rfloor}\binom{n+r-1}{k}\binom{2n+2r-2k-2}{2n+r-1}(-1)^k=2^{r-1}\binom{n+r-1}{n};\ n,r\in\mathbb{N}; \tag{3.5}$$

$$\sum_k\binom{n}{k}\binom{r-sk}{n}(-1)^k=s^n\quad r,s\in\mathbb{R};\qquad \textbf{\textit{(b)}}\sum_{k=0}^{t}\frac{\binom{t+v-r-s}{t-k-r}}{\binom{t+v}{t-k}}=\frac{\frac{t+v+1}{r+s+1}}{\binom{r+s}{r}},\quad s\ge v;\ t+v\ge r+s; \tag{3.6}$$

$$\sum_k\frac{\binom{m}{k}}{\binom{n}{k}}=\frac{n+1}{n-m+1}\quad n\ge m;\qquad \textbf{\textit{(b)}}\sum_{k=1}^{s}\frac{\binom{k+x}{k}}{\binom{k+y}{k}}=\sum_{k=1}^{s}\frac{(x+1)^{\overline{k}}}{(y+1)^{\overline{k}}}=\frac{(s+1+y)(s+1+x)^{\underline{s+1}}}{(x+1-y)(s+1+y)^{\underline{s+1}}}-\frac{x+1}{x-y+1} \tag{3.7}$$

$$\sum_{k=0}^{s}\frac{\Gamma(x+k)}{\Gamma(y+k)}=\frac{y+s}{x+1-y}\frac{\Gamma(s+1+x)}{\Gamma(s+1+y)}-\frac{y-1}{x+1-y}\frac{\Gamma(x)}{\Gamma(y)}.\ \text{same as (3.7)(b)} \tag{3.8}$$

$$\sum_{k=0}^{s}\frac{\binom{a+n}{k}}{\binom{b+n}{k}}=\frac{(b+n-s)(a+n)^{\underline{s+1}}}{(a-1-b)(b+n)^{\underline{s+1}}}-\frac{b+n+1}{a-1-b} \tag{3.9}$$

$$\sum_k\binom{r+tk}{k}\binom{s-tk}{n-k}\frac{r}{r+tk}=\binom{r+s}{n};\qquad \sum_k\binom{n}{k}(r+tk)^{k-1}(s-tk)^{n-k}=\frac{(r+s)^n}{r}\quad r,s\in\mathbb{R} \tag{3.10}$$

$$\sum_{k\ge 0}\binom{r+tk}{k}\binom{s-tk}{n-k}=\sum_{k\ge 0}\binom{p-k}{n-k}t^k=\begin{cases}t^{p+1}(t-1)^{n-p-1} & \text{integer } p<n\\ (1-t^{n+1})/(1-t) & p=n\\ \text{no closed form when } p>n \text{ or when } p\notin\mathbb{N}\end{cases}\qquad p\equiv r+s \tag{3.11}$$

$$\sum_{k\ge 0}\binom{m}{k}\binom{n-k}{n-k-l}(t-1)^k=\sum_{k\ge 0}\binom{m}{k}\binom{n-m}{n-k-l}t^k\quad l,m,n\in\mathbb{Z}_{\ge 0}. \tag{3.12}$$

$$\sum_k\binom{-1/2}{k}\binom{1/2}{n+1-k}(-1)^k=2^{-n}\binom{n}{\lfloor n/2\rfloor}.\qquad \sum_k\binom{2n}{k}\binom{2n+2}{k}(-1)^k=\frac{(-1)^n}{n+1}\binom{2n}{n}. \tag{3.13}$$

$$\sum_{i\ge 0}\binom{-1/2}{i}\frac{1}{2i+1}\binom{-1/2}{k-i-1}\frac{1}{2k-2i-1}=\frac{(-1)^{k+1}2^{2k-1}}{k^2\binom{2k}{k}}\quad \textbf{\textit{(b)}}\sum_{k=0}^{m}\binom{n+m-2k}{n-k}\binom{2k}{k}\frac{1}{k+1}=\binom{n+m+1}{m}. \tag{3.14}$$

$$\sum_{j\ge0}\binom{m}{j}\binom{j/2}{n}=(-1)^{n-1}\frac{m}{n}2^{m-2n}\binom{2n-m-1}{n-1},\qquad \textbf{(b)}\sum_{j\ge0}\binom{m}{j}\binom{j/2}{n}(-1)^j=(-1)^n\frac{m}{n}2^{m-2n}\binom{2n-m-1}{n-1},\ n\ge m \qquad (3.15)$$

$$\sum_k\binom{x+1}{k}\binom{x+a-1}{k+a-1}\binom{x-k}{n-k}=\binom{x+a-1}{n+a-1}\binom{x+n+a}{n}\qquad \textbf{(b)}\sum_k\binom{x+y+k}{k}\binom{y}{a-k}\binom{x}{b-k}=\binom{x+a}{b}\binom{y+b}{a} \qquad (3.16)$$

$$\sum_{k=0}^{n}(a+rk)q^k=\frac{a-(a+nr)q^{n+1}}{1-q}+\frac{rq(1-q^n)}{(1-q)^2};\qquad \sum_{k=0}^{n}kq^k=\frac{q}{(1-q)^2}\left[1-(n+1)q^n+nq^{n+1}\right] \qquad (4.1)$$

$$\sum_{k=0}^{n}k^2q^k=\frac{q}{(1-q)^3}\Big[1+q-(n+1)^2q^n+\big(2n^2+2n-1\big)q^{n+1}-n^2q^{n+2}\Big];\quad S_r^{(n)}(q)\equiv\sum_{k=1}^{n}k^rq^k : S_{r+1}^{(n)}(q)=qD_qS_r^{(n)}(q). \qquad (4.2)$$

$$\sum_{k=0}^{n}k(-1)^k=(-1)^n\left\lceil\frac{n}{2}\right\rceil;\qquad \sum_{k=0}^{n}k^2(-1)^k=\frac{(-1)^n}{2}n(n+1);\qquad \sum_{k=a}^{n}k^{\underline{m}}=\frac{k^{\underline{m+1}}}{m+1}\Big|_{k=a}^{n+1},\quad m\ne-1,\ a\ge0; \qquad (4.3)$$

$$\sum_{k=0}^{n}k^{\underline{2}}q^k=\frac{q^2}{(1-q)^3}\Big[2-n(n+1)q^{n-1}+2\big(n^2-1\big)q^n-n(n-1)q^{n+1}\Big];\quad \textbf{(b)}T_r(q)=\sum_k k^{\underline{r}}q^k=\frac{r!q^r}{(1-q)^{r+1}}. \qquad (4.4)$$

$$H_n=\ln n+\gamma+\frac{1}{2n}-\frac{1}{12n^2}+\frac{1}{120n^4}+O(n^{-6}) \qquad (4.5)$$

$$\sum_{k=1}^{n}H_k=(n+1)H_n-n=(n+1)(H_{n+1}-1)\quad \textbf{(b)}\sum_{k=1}^{n}\binom{k}{m}H_k=\binom{n+1}{m+1}\left[H_{n+1}-\frac{1}{m+1}\right],\quad \textbf{(c)}\sum_{k=1}^{n}\frac{H_k}{k}=\frac12\left[H_n^2+H_n^{(2)}\right] \qquad (4.6)$$

$$\sum_{k=1}^{n}\frac{H_k}{k+1}=\frac12\left[H_{n+1}^2-H_{n+1}^{(2)}\right];\quad \textbf{(b)}\sum_{k=2}^{n}\frac{H_k}{k-1}=\frac12\left[H_{n-1}^2+H_{n-1}^{(2)}\right]+1-\frac1n;\quad \textbf{(c)}\sum_{k=1}^{n}H_k^2=(n+1)H_n^2-(2n+1)H_n+2n. \qquad (4.7)$$

$$\sum_{k=1}^{2n}(-1)^kH_k=H_n/2;\quad \textbf{(b)}\sum_{k=1}^{n}H_kH_{n+1-k}=(n+2)(H_{n+1}^2-H_{n+1}^{(2)})-2(n+1)H_n+2n. \qquad (4.8)$$

$$\sum_{k=0}^{n}k^m=\frac{1}{1+m}\sum_{k=0}^{m}\binom{m+1}{k}(n+1)^{m-k+1}B_k\quad=\frac{B_{m+1}(n+1)-B_{m+1}}{1+m}=\frac{n^{m+1}}{m+1}\left(1+O\left(n^{-1}\right)\right). \qquad (4.9)$$

B_k, $B_k(\cdot)$ are Bernoulli numbers, (6.12). and polynomials. $\zeta(s,m)$ is the Hurwitz function.

$$\sum_{k=0}^{n}F_kF_{n-k}=\frac15(nL_n-F_n)=\frac15[2nF_{n-1}+(n-1)F_n].\qquad \textbf{(b)}\quad \sum_{k=0}^{n}kF_kF_{n-k}=\frac{n}{10}[2nF_{n-1}+(n-1)F_n]. \qquad (4.10)$$

$$\sum_{n\ge1}\prod_{i=1}^{n}\frac{q}{b+i}=qe^q\sum_{k\ge0}\frac{(-q)^k}{k!}\frac{1}{b+k+1},\quad \sum_{n\ge1}n\prod_{i=1}^{n}\frac{q}{b+i}=qe^q\sum_{k\ge0}\frac{(-q)^k}{k!}\frac{q+k+1}{b+k+1},\quad b\notin-\mathbb{N} \qquad (5.1)$$

$$\sum_{n\ge1}n^2\prod_{i=1}^{n}\frac{q}{b+i}=qe^q\sum_{k\ge0}\frac{(-q)^k}{k!}\cdot\frac{q+(q+k+1)^2}{b+k+1},\qquad b\notin-\mathbb{N}\quad \text{If } b=0 \text{ the sum is } e^q(1+q)q. \qquad (5.2)$$

$$\sum_{\substack{k\ne l\\k=0}}^{n}\frac{1}{k-l}\left[(-1)^l\binom{n}{l}+(-1)^k\binom{n}{k}\right]=0,\quad 0\le l\le n;\qquad \sum_{n\ge0}\prod_{i=1}^{n}\frac{q}{r+i}=\frac{r!}{q^r}\big(e^q-e_{r-1}(q)\big),\quad r\in\mathbb{N}; \qquad (5.3)$$

$$\sum_{j\ge0}\frac{(qz)^j}{(z+1)(z+q)\cdots(z+q^j)}=1,\quad q,z\in\mathbb{C},\ |q|>1\ (\text{or } |q|=1,|z|<1); \qquad (5.4)$$

$$\sum_{j=1}^{n}\frac{a_j^k}{\prod_{\substack{i=1\\i\ne j}}^{n}(a_j-a_i)}=\begin{cases}0 & k<n-1\\ 1 & k=n-1\\ h_{k-n+1}(\boldsymbol{a}) & k\ge n\end{cases}\left|\begin{array}{l}\text{distinct } a_i\in\mathbb{C};\\ h_r(\boldsymbol{a})=\sum\limits_{\substack{i_j\ge0\\ \Sigma i_j=r}}\prod_{j=1}^{n}a_j^{i_j}\ \text{symm. poly}\end{array}\right.\implies\sum_{j=1}^{n}\prod_{\substack{k\ne j\\k=1}}^{n}\frac{z-a_j}{a_k-a_j}=1. \qquad (5.5)$$

$\sigma_k(t_1,\ldots t_n)=[z^{n-k}](z+t_1)\cdots(z+t_n)$, the kth elementary symmetric polynomial in the n variables t_j, $1\le k\le n$.

Let $S_m\equiv\sum_{i=1}^{n}t_i^m$, then $S_m=\sum\limits_{i_1+2i_2+\cdots+ni_n=m}(-1)^{i_2+i_4+i_6+\cdots}\dfrac{(i_1+i_2+\cdots+i_n-1)!m}{i_1!i_{2!}\cdots i_n!}\sigma_1^{i_1}\sigma_2^{i_2}\cdots\sigma_n^{i_n}$, Waring formula (5.6)

$$\sum_{r\ge0}\frac{(uv)^rz^{\binom{r}{2}}}{\prod_{i=0}^{r}(1-vz^i)}=\sum_{i\ge0}v^i\prod_{j=0}^{i-1}\big(1+uz^j\big);\ \text{ viz. (5.8)(a).}\qquad \text{The case } k=n-1 \text{ in (5.5) is Good's identity..} \qquad (5.7)$$

$$\sum_{n\ge0}\frac{u^n}{\prod_{i=1}^{n}(1-q^i)}=\frac{1}{\prod_{i\ge0}(1-uq^i)},\quad \sum_{r\ge0}\frac{u^rz^{\binom{r}{2}}}{\prod_{i=1}^{r}(1-z^i)}\overset{(a)}{=}1+u\sum_{i\ge0}z^i\prod_{j=0}^{i-1}\big(1+uz^j\big)=\prod_{k\ge0}\big(1+uz^k\big),\ \text{Euler identities} \qquad (5.8)$$

$$\frac{(uz)_\infty}{(z)_\infty}=\sum_{n\ge0}\frac{(u)_nz^n}{(q)_n},\ \text{ with } (a)_n=(a;q)_n\equiv\prod_{i=0}^{n-1}\big(1-aq^i\big);\quad (a;q)_0=1;\quad (a;q)_t=\frac{(a;q)_\infty}{(aq^t;q)_\infty},\ \begin{array}{l}\text{Cauchy identity}\\ q-\text{Pochhammer symbol}\end{array} \qquad (5.9)$$

$$\sum_{n\geq 0}\frac{(a)_n(b)_n}{(c)_n(q)_n}t^n=\frac{(b)_\infty(at)_\infty}{(c)_\infty(t)_\infty}\sum_{k\geq 0}\frac{(c/b)_k(t)_k}{(at)_k(q)_k}b^k \qquad \text{Heine's transformation} \tag{5.10}$$

$$\sum_{k=1}^{n}a_k\Delta b_k=a_kb_k\Big|_1^{n+1}-\sum_{k=1}^{n}\left(\Delta a_k\right)b_{k+1} \text{ summation by parts.} \tag{6.1}$$

$$\sum_{1\leq i,j\leq n}\min\left(x_i,x_j\right)=2\left(\sum_{i=1}^{n}ix_{(i)}-\sum_{i=1}^{n}x_i\right),\ \text{where } x_{(1)}\geq x_{(2)}\geq\cdots\geq x_{(n)}. \tag{6.2}$$

$$\sum_{m=1}^{M}\sum_{n=1}^{N}\min(m,n)=\frac{1}{6}N(N+1)(3M-N+1). \qquad \sum_{m=1}^{M}\sum_{n=1}^{N}\max(m,n)=\frac{1}{6}N\left(N^{2-1}\right)+\frac{1}{2}MN(M+1). \quad M\geq N \tag{6.3}$$

$$\int e^{-\lambda t}t^n dt=-\frac{e^{-\lambda t}}{\lambda^{n+1}}n!\sum_{i=0}^{n}\frac{(t\lambda)^i}{i!};\quad \int_{t=0}^{\infty}e^{-\lambda t}(a+t)^j dt=\frac{j!}{\lambda^{j+1}}\sum_{i=0}^{j}\frac{(a\lambda)^i}{i!}; \tag{6.4}$$

$$\int_{t=0}^{1}t^{z-1}(1-t)^{w-1}dt=\int_{u=0}^{\infty}\frac{u^{z-1}}{(1+u)^{w+z}}du=B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(w+z)}, \tag{6.5}$$

$$\int_{-\infty}^{\infty}x^{2j}e^{-ax^2}dx=\frac{1}{a^{(2j+1)/2}}\Gamma\left(j+\frac{1}{2}\right)=\frac{\sqrt{\pi}(2j)!}{\sqrt{a}(4a)^j j!} \qquad \textbf{(b)}\quad \mathbb{E}\left[\left(\mathbb{N}(0,\sigma^2)\right)^{2j}\right]=\sigma^{2j}\frac{(2j)!}{2^j j!}=\sigma^{2j}(2j-1)!! \tag{6.6}$$

$$\int_{-\infty}^{\infty}e^{-ax^2}dx=\sqrt{\pi/a},\quad \textbf{(b)}\int_{-\infty}^{\infty}e^{-ax^2+bx}dx=\sqrt{\pi/a}\,e^{b^2/4a},\quad \textbf{(c)}\int_{0}^{\infty}e^{-ax^2+bx}dx=\sqrt{\pi/a}\,e^{b^2/4a}\Phi\left(b/\sqrt{2a}\right). \tag{6.7}$$

$$\int_{0}^{\infty}xe^{-ax^2+bx}dx=\frac{1}{2a}+\frac{b}{a}\sqrt{\pi/a}\,e^{b^2/4a}\Phi\left(b/\sqrt{2a}\right),\quad \textbf{(b)}\int_{-\infty}^{\infty}xe^{-ax^2+bx}dx=\frac{b}{a}\sqrt{\pi/a}\,e^{b^2/4a} \tag{6.8}$$

$$\int_{0}^{\infty}x^{2k}e^{-ax^2}dx=\frac{(2k)!\sqrt{\pi/a}}{2(4a)^k k!},\quad \textbf{(b)}\int_{0}^{\infty}x^{2k+1}e^{-ax^2}dx=\frac{k!}{2a^{k+1}} \qquad \text{see (6.6)} \tag{6.9}$$

$$\sum_{k=1}^{n}f(k)=\int_{1}^{n}f(x)+1/2[f(n)+f(1)]+\sum_{j=1}^{m}\frac{B_{2j}}{(2j)!}\left[f^{[2j-1]}(n)-f^{[2j-1]}(1)\right]+\frac{1}{(2m+1)!}\int_{1}^{n}B_{2m+1}(\{x\})f^{[2m+1]}(x)dx. \tag{6.10}$$

$$\left\lfloor\frac{n}{2}\right\rfloor+\left\lceil\frac{n}{2}\right\rceil=n: \qquad \left\lfloor\frac{n+1}{2}\right\rfloor=\left\lceil\frac{n}{2}\right\rceil: \qquad \left\lfloor\frac{n-1}{2}\right\rfloor=\left\lceil\frac{n}{2}\right\rceil-1 \tag{6.11}$$

$$B_{k\geq 0}: 1,-1/2,1/6,\ \text{(even)}\ -1/30,1/42,-1/30,5/66,\quad -691/2730,7/6,-3617/510,43867/798,-174611/330 \tag{6.12}$$

$$\zeta(1-s)=\frac{2}{(2\pi)^s}\cos\left(\frac{s\pi}{2}\right)\Gamma(s)\zeta(s);\quad \zeta(1-n)=(-1)^{n+1}B_n/n,\ n\geq 1. \tag{6.13}$$

$$\sum_{k=1}^{m}k^{-s}=\zeta(s)-\zeta(s,m+1);\quad v\zeta(v+1,m+1)\sim\sum_{j\geq 0}\frac{v^{\bar{j}}B_j}{j!m^{j+v}}=\frac{1}{m^v}-\frac{v}{2m^{v+1}}+\frac{v(v+1)}{12m^{v+2}}+\cdots;\quad m\to\infty\ \text{(see (6.12))} \tag{6.14}$$

$$\ln(1+x)=-\sum_{r\geq 1}\frac{(-x)^r}{r}=\sum_{r\geq 1}\frac{1}{r}x^r\left(1-x^r\right),\quad |x|<1.\quad \ln(x)\overset{(x\geq 1/2)}{=}\sum_{k\geq 1}\frac{(x-1)^k}{kx^k}\overset{(x>0)}{=}\sum_{k\geq 0}\frac{1}{2k+1}\left(\frac{x^2-1}{x^2+1}\right)^{2k+1}. \tag{6.15}$$

$$1-\Phi(x)\sim\frac{\phi(x)}{x}\left(1-x^{-2}+3x^{-4}\cdots\right)\ \text{(A\&S, 26.2.12)},\quad \operatorname{erf}(x)=2\Phi\left(x\sqrt{2}\right)-1.\qquad \Psi(1,a)=\frac{\pi^2}{6}-H_{a-1}^{(2)}. \tag{6.16}$$

INEQUALITIES

Markov: $$\frac{\mathbb{E}|X|^r-a^r}{a.s.\sup|X|^r}\leq\mathbb{P}\{|X|\geq a\}\leq\mathbb{E}\left[\left(\frac{|X|}{a}\right)^n\right] \qquad r,\ n\geq 0. \tag{7.1}$$

This is the best possible when the only information is that X is non-negative and with finite mean. When it is also known that X is unimodal about M, then $\mathbb{P}[|X-M|\geq a]<\left(\frac{n}{n+1}\right)^n\mathbb{E}\left[\left|\frac{X-M}{a}\right|^n\right]$. □

Inequality (7.1) for integer rv's and $n=1$ becomes: $\mathbb{P}(X>0)\leq\mathbb{E}[X]$. also for any rv $\mathbb{P}(X=0)\leq\sigma^2/\mathbb{E}[X^2]$. (7.2)

Bienaymè–Chebyshev: $\mathbb{P}\{|X|\geq t\}\leq t^{-2}\mathbb{E}[X^2]$ same as the RHS of (7.1) for $n=2$. (7.3)

Chebyshev –Cantelli: $\mathbb{P}\{X-\mu\geq t\}\leq\dfrac{\sigma^2}{\sigma^2+t^2}$ (7.4)

Both (7.1) and (7.3) are special cases of the following, more general inequality (Loéve, “Probability Theory” Vol I, p. 159): Let X be an arbitrary r.v. and g on R be a nonnegative Borel Function. If g is even and nondecreasing on $[0,+\infty)$ then, for every $a \geq 0$

Loéve: $$\frac{\mathbb{E}[g(X)]-g(a)}{a.s.\sup g(X)} \leq \mathbb{P}[|X| \geq a] \leq \frac{\mathbb{E}[g(X)]}{g(a)} \tag{7.5}$$

If g is nondecreasing on R, then the middle term is replaced by $\mathbb{P}[X \geq a]$, where a is an arbitrary number. □

Let $X_1, X_2, \ldots, X_n$ be independent rv's with $\mathbb{E}X_i^2 < \infty$, and S_k the sum of the first k X_i's. Then, for any $\varepsilon > 0$

Kolmogorov: $$\mathbb{P}\Big(\max_{1\leq k\leq n} \big|S_k - \mathbb{E}[S_k]\big| \geq \varepsilon\Big) \leq \frac{\sigma_n^2}{\varepsilon^2} \qquad \sigma_n^2 \equiv V[S_n]. \tag{7.6}$$

This also holds when S_n is a martingale. For $n=1$ (7.6) reduces (7.3). □

Let X_i, $1 \leq i \leq n$ be i.i.d. bounded rv's, $0 \leq X_i \leq 1$, $\mu = \mathbb{E}[S = \Sigma X_i]/n$, then, for $0 < t < 1-\mu$:

Hoeffding: $$\mathbb{P}[S - n\mu \geq nt] \leq \left[\left(\frac{\mu}{\mu+t}\right)^{\mu+t}\left(\frac{1-\mu}{1-\mu-t}\right)^{1-\mu-t}\right]^n \leq \exp\big(-nt^2 g(\mu)\big) \leq \exp\big(-2nt^2\big), \tag{7.7}$$

where

$$g(\mu) = \begin{cases} \frac{1}{1-2\mu}\log\left(\frac{1-\mu}{\mu}\right) & 0 < \mu < \frac{1}{2} \\ \frac{1}{2\mu(1-\mu)} & \frac{1}{2} \leq \mu < 1 \end{cases} \qquad \text{Note}: g(\mu) \geq 2, \quad \forall \mu.$$

For $t > 1-\mu$ the first bound in the right-hand-side in (7.7) is 0, and for $t = 1-\mu$ it equals μ^n. (Hoeffding, *J. Am. Stat. Ass.* **58**, 13–30, 1963). The right-most inequality is often named after **Chernoff**. □

The combinatorial **Bonferroni** inequalities are easily adapted probability measures (Comtet, pp. 192–3). Let $A = \{A_i, 1 \leq i \leq n\}$ be a set of events in some probability space Ω, associated with a measure $\mathbb{P}$, and define the probabilities S_k as the sum of probabilities of occurrence of the intersections of all k-strong subsets of these events: $S_0 \equiv \mathbb{P}(\Omega)$, $\quad S_k = \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} \mathbb{P}\left(A_{i_1}A_{i_2}\cdots A_{i_k}\right)$. Then the sum $\sum_{k=1}^n (-1)^{k-1} S_k$ satisfies the *alternating inequalities*:

Bonferroni: $$(-1)^k\left[\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) + \sum_{j=1}^k (-1)^j S_j\right] \geq 0 \qquad (-1)^{k+1}\left[\mathbb{P}\left(\bigcap_{i=1}^n \overline{A}_i\right) + \sum_{j=0}^k (-1)^{j+1} S_j\right] \geq 0 \quad 1 \leq k \leq n \tag{7.8}$$

For $k=1$ the above reduces **Boole**'s inequality. □

Let μ_i be moments of X, and $L(s)$ its LST. $L(s)$ is absolutely monotone, and

Laplace-Stieltjes Transform: $$\sum_{i=0}^{2n-1}\frac{\mu_i(-s)^i}{i!} \leq L(s) \leq \sum_{i=0}^{2n}\frac{\mu_i(-s)^i}{i!} \quad \forall n \geq 1. \tag{7.9}$$

Jensen: $\mathbb{E}[u(X)] \geq u(\mathbb{E}[X])$ when $u(\cdot)$ is a convex function. □ (7.10)

Chebyshev: $$n\sum_{k=1}^n a_k b_k \geq \left(\sum_{k=1}^n a_k\right)\left(\sum_{k=1}^n b_k\right) \quad \text{where } \{a_i\} \text{ and } \{b_i\} \text{ are monotonic, both increasing or decreasing,} \tag{7.11}$$

If the senses of monotonicity are opposite, reverse the inequality sign. □

Hölder: $$\sum_{k=1}^n |a_k b_k| \leq \left(\sum_{k=1}^n |a_k|^p\right)^{1/p}\left(\sum_{k=1}^n |b_k|^q\right)^{1/q} \quad \text{for } p>1,\ q>1, \text{ and } p^{-1}+q^{-1}=1. \tag{7.12}$$

The **Cauchy-Bunyakovsky-Schwartz (CBS)** inequality is obtained for $p=q=2$. The analogue holds for integrals. □

The CBS inequality also leads the following (in Vitter & Krishnan, *JACM* **43**, #5 p.785):

Amit-Miller: $$\left(\sum_{k=1}^n |p_k - r_k|\right)^2 \leq 2\sum_{k=1}^n p_k \ln(p_k/r_k), \quad \text{for any two } n \text{ term PMFs } \boldsymbol{p} \text{ and } \boldsymbol{r} \tag{7.13}$$

Gibbs: $$\sum_{k=1}^n p_k \log(p_k/r_k) \geq 0, \quad \text{for any two } n \text{ term PMFs } \boldsymbol{p} \text{ and } \boldsymbol{r} \text{ (with equality if the PMFs coincide).} \tag{7.14}$$

Minkowski-I: $$\left(\sum_{k=1}^n (a_k+b_k)^p\right)^{1/p} \leq \left(\sum_{k=1}^n a_k^p\right)^{1/p} + \left(\sum_{k=1}^n b_k^p\right)^{1/p}. \quad \text{for } p>1 \text{ and } a_k,\ b_k > 0, \forall k. \tag{7.15}$$

The analogue holds for integrals. □

Minkowski-II: $$\prod_{k=1}^n (1+a_k) \geq \left(1+\left(\prod_{k=1}^n a_k\right)^{1/n}\right)^n. \quad \text{for } a_k \geq 0 \text{ for all } k \text{ and integer } n. \tag{7.16}$$

Carleman: $\sum_{k=1}^{n}\left(a_1a_2\cdots a_k\right)^{1/k} \le e\sum_{k=1}^{n} a_k$ for $a_k \ge 0$ for all k, and an integer n. (7.17)

On the interval $[a,b]$ let $f(\cdot)$ be real and continuous and let there be a family (not necessarily finite) of real and continuous orthonormal functions, $\{\psi_r(\cdot)\}$, then

Bessel: $\sum_r \left[\int_a^b f(x)\psi_r(x)dx\right]^2 \le \int_a^b f^2(x)dx,$ The conditions on f, which may be complex, can be relaxed. (7.18)

Hilbert: $\sum_{m,n}\frac{a_mb_n}{m+n} < \frac{\pi}{\sin(\pi/p)}A^{1/p}B^{1/q},$ when $p,q>1$, $1/p+1/q=1$, $\Sigma_m a_m^p \le A$, $\Sigma_n b_n^q \le B$. (7.19)

unless one of the series vanishes identically. (More on this in Hardy, Littlewood and Pólya, Chapter IX). □

interpolation: $\frac{n}{n+1}\,\frac{m}{m-1}\left(\frac{n-1}{n}\right)^{n-m+1} \le 1, \quad 1<m<n.$ $f(x)=x^p(1-x)^{1-p}, \quad p\in(0,1)$ is maximized at $x=p$. (7.20)

The wedge inequality: $|(a\vee b)-(c\vee d)| \le |a-c|\vee|b-d|$ $a,b,c,d\in\mathbb{R}$; $\vee$: selection of the maximum : (7.21)

The inequality holds also for functions instead of numbers, with $|\cdot|$ replaced by any p-norm, $p>0$. □

The ratio inequality: $\sum_{i=1}^{n} a_i \Big/ \sum_{j=1}^{n} b_j \le \max_k \frac{a_k}{b_k}$ $a_k,\ b_k\in\mathbb{R}^+,$ (7.22)

Let A, G and H denote the arithmetic, geometric and harmonic means of $\{a\}=\{a_i, 1\le i\le n\}$, given by

The Three Means: $A=\sum a_i/n, \quad G=[\prod a_i]^{1/n}, \quad H=[\sum a_i^{-1}/n]^{-1}.$ (7.23)

The standard relation $A\ge G\ge H$ can be generalized as follows: Let $M_t\{a\}\equiv(\frac{1}{n}\sum a_i^t)^{1/t}$, then

Power-means (Arnon): $t_1>t_2 \Longrightarrow M_{t_1}\{a\}\ge M_{t_2}\{a\}.$ Known as **Lyapunov inequality** in probability. (7.24)

A, G, H correspond to $t=1,0,-1$. As $t\to\pm\infty$, $M_t\{a\}$ approaches $\max\{a\}$ and $\min\{a\}$ resp. The power-means ineq. can be further generalized with weights: Let $\boldsymbol{p}$ a PMF, and $\mathbb{E}_t^p\{a\}\equiv(\sum p_ia_i^t)^{1/t}$, then

Weighted power-means: $t_1>t_2 \Longrightarrow \mathbb{E}_{t_1}^p\{a\}\ge\mathbb{E}_{t_2}^p\{a\}.$ □ (7.25)

Let $N_t\{a\}\equiv(\sum a_i^t)^{1/t}$, with all $a_i>0$, then

Norm monotonicity: $t_1>t_2 \Longrightarrow N_{t_1}\{a\}\le N_{t_2}\{a\}.$ (7.26)

The Jackknife inequality (Efron & Stein *Ann. Statist.* **9**, 586–596, 1981).
Let $\{X_i\}$, $1\le i\le n$ be i.i.d and H a symmetric measurable function over $n-1$-dimensional vectors.
Define $S_i=H(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n)$, and $S=\Sigma_{i=1}^n S_i/n$. Then

The Jackknife inequality: $V\Big[H(X_1,\ldots,X_{n-1})\Big] \le \mathbb{E}\left[\sum_{i=1}^{n}(S_i-S)^2\right].$ (7.27)

Shachnai: $0<q<1, a_i\ge0, \quad \sum_{i=1}^{n} a_iq^i\sum_{j=1}^{i-1} a_j \le \frac{1+3q}{2(1-q)}\sum_{i=1}^{n} a_i^2q^i$ (7.28)

Shown in A. Bar-Noy, M. Bellare, M. Halldorsson, H. Shachnai, T. Tamir, *Inf. and Comput.*, **140**, 1998, 183–202.

$\frac{4^n}{2\sqrt{n}} \le \binom{2n}{n} \le \frac{4^n}{\sqrt{3n+1}};$ ***(b)*** $\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left\{\frac{n^n}{k^k\,(n-k)^{n-k}}, \quad \left(\frac{en}{k}\right)^k\right\}.$ (7.29)

Logarithm: $\lim_{x\to\infty} x^{-a}\log x=0,$ ***(b)*** $\lim_{x\to0} x^a\log x=0, \quad \Re a>0$ ***(c)*** $\ln x\le n(x^{1/n}-1),\ x>0, n\in\mathbb{Z}_+$ (7.30)

$\frac{x}{x+1} < \ln(1+x) < x, \quad 0\ne x>-1,$ ***(b)*** $x<-\ln(1-x)<\frac{x}{1-x}, \quad 0\ne x<1,$ ***(c)*** $\ln x\le x-1,\ x>0$ (7.31)

$|\log(1+z)|\le-\log(1-|z|). \quad |z|<1.$ ***(b)*** $\left(1+\frac{1}{x}\right)^x$ $x>0$ is monotonic incr. (7.32)

Exponent: $e^{-\frac{x}{1-x}}<1-x<e^{-x},\ x\in[0,1),$ ***(b)*** $\frac{1}{1-x}>(x<1)\ e^x>1+x,$ ***(c)*** $\frac{x}{1+x}<1-e^{-x}<x,\ (x>-1).$ (7.33)

$1+x>e^{\frac{x}{1+x}}\ x>-1,$ ***(b)*** $e^x>\left(1+\frac{x}{y}\right)^y>e^{\frac{xy}{x+y}} \quad x,y>0$ ***(c)*** $|e^z-1|\le e^{|z|}-1\le|z|e^{|z|}$ all z. (7.34)

Let $m_a=\mathbb{E}[|X-\mathbb{E}X|^a]$, $a\in\mathbb{R}$, then $m_a^{1/a}$ is a monotonic increasing function in a. See (7.24) (8.1)

$\int_{p=0}^{1} F_p^r dp=\mathbb{E}(X^r),$ where F_p is the $p-$ fractile of the rv $X:\mathbb{P}\big(X\le F_p\big)=p.$ (8.2)

$\sum_{i\ge1}\int_{t=0}^{u}[1-G(u-t)]g^{*i}(t)dt=G(u), \quad g(t)=\frac{dG(t)}{dt};$ $1-\Phi(x)\sim\frac{\phi(x)}{x}\big(1-x^{-2}+3x^{-4}\cdots\big)$ (A & S, 26.2.12). (8.3)

For the $M/G/1$ queue: PGF of customers in system, mean value and variance, where $S(t)$ is the LST of the service time distribution and σ_r is $\mathbb{E}(S^r)$:

$$g(z) = p_0 \frac{\tilde{S}(z)(z-1)}{z-\tilde{S}(z)}, \quad \tilde{S}(z) \equiv S(\lambda - \lambda z), \quad p_0 = 1 - \lambda \mathbb{E}(S) = 1-\rho, \quad \mathbb{E}(X) = \rho + \frac{\lambda^2 \sigma_2}{2(1-\rho)}, \quad \mathbb{E}(W) = \frac{\lambda \sigma_2}{2(1-\rho)} \tag{8.4}$$

$$\mathbb{E}(X^2) = \mathbb{E}(X) + \frac{\lambda^3 \sigma_3 + 3\lambda^2 \sigma_2}{3(1-\rho)} + \frac{\lambda^4 \sigma_2^2}{2(1-\rho)^2}, \quad W(s) = \frac{s(1-\rho)}{\lambda S(s) - \lambda + s}.$$

For the $M/G/\infty$ queue: if arrivals are in batches where the batch size has the PGF $A(z)$, and $F(\cdot)$ is the service time CDF, the number in queue has the Poisson distribution only when $A(z) = z$:
The Expected length of a bp and of the number of customers served in a bp are given by

$$\mathbb{E}[B] = \frac{e^{\lambda \xi} - 1}{\lambda}, \quad \mathbb{E}[N] = \alpha e^{\lambda \xi}, \quad \alpha = A'(1), \quad \xi \equiv \int_{t=0}^{\infty} (1 - A(F(t)))dt. \tag{8.5}$$

When $A(z) = z$, $\xi = \mathbb{E}[S]$, $\mathbb{E}[B] = (e^{\rho} - 1)/\lambda$, $\mathbb{E}[N] = e^{\rho}$. (Shanbhag, J. Appl. Prob. **3**, 274–279 (1966)).

Volume of an n-dimensional ball of radius R: $\quad V_n(R) = 2^{\lceil n/2 \rceil} \pi^{\lfloor n/2 \rfloor} R^n / n!! = \pi^{n/2} R^n / \Gamma(n/2+1). \quad A_n(R) = dV_n(R)/dR$ (8.6)

Number of grid points in corner of d-dimensional cube of side m: $\binom{m+d}{m}$ (8.7)

Moments of $N(a, \sigma^2)$: $\quad \mathbb{E}(X-a)^{2r} = (2r)!\sigma^{2r}/2^r\Gamma(r+1). \quad \mathbb{E}|X-a|^r = 2^{r/2}\Gamma((r+1)/2)\sigma^r/\sqrt{\pi}$, odd r (8.8)

The eigenvalues of an n-circulant with first column $\underline{c}$ are given by

$$\underline{\lambda} = \sqrt{n}\boldsymbol{W}\underline{c}, \text{ where } W_{jk} = \exp\left(i\frac{2\pi}{n}jk\right). \tag{8.9}$$

GENERIC EQUATIONS

The general linear differential equation of first order:

$$y'(x) + h(x)y(x) = g(x), \tag{9.1e}$$

with $h(\cdot)$ and $g(\cdot)$ known functions, and a prescribed boundary value $y_0 = y(x_0)$, has the general solution

$$y(x) = e^{-\int h dx}\left(C + \int g e^{\int h dx} dx\right), \quad \square \tag{9.1s}$$

where the constant C is determined from the boundary value.

The Bernoulli equation is $y' + p(x)y = q(x)y^n$; the substitution $v = y^{1-n}$ linearizes it to $v' = (n-1)[p(x)v - q(x)]$.

Homogeneous Ricatti equation with constant coefficients, aka logistic equation (a particular case of Bernoulli equation):

$$f'(x) = af^2(x) + bf(x), \quad f(0) = \alpha, \text{ where } a \text{ and } b \text{ are constants, has the solution} \tag{9.2e}$$

$$f(x) = \frac{\alpha e^{bx}}{1 - a\alpha b^{-1}(e^{bx} - 1)}, \qquad \text{(Goulden and Jackson, 1984).} \tag{9.2s}$$

The general linear first order difference equation

$$a_{t+1} = \alpha_t + \beta_t a_t; \quad t \geq m, \text{ with } a_m, \{\alpha_k\}, \{\beta_k\} \text{ known,} \tag{9.3e}$$

has the solution

$$a_t = \sum_{j=m}^{t-1} \alpha_j \prod_{i=j+1}^{t-1} \beta_i + a_m \prod_{i=m}^{t-1} \beta_i \qquad t \geq m \tag{9.3s}$$

Special cases: Constant α and β

$$a_{t+1} = \alpha + \beta a_t; \quad t \geq m \quad \Longrightarrow \quad a_t = \frac{\alpha}{1-\beta} + \beta^{t-m}\left(a_m - \frac{\alpha}{1-\beta}\right), \quad t \geq m. \tag{9.3c}$$

Pure linear β_t, constant α_t:

$$a_{i+1} = b + ica_i, \quad i \geq 1, \quad a_1 \text{ given} \tag{9.5e}$$

$$a_i = (i-1)!c^{i-1}\left[a_1 + b(e_{i-1}(1/c) - 1)\right], \quad i \geq 1. \qquad e_r(u) - \text{ the incomplete exponential function.} \tag{9.5s}$$

Functional equation: $a(x)$ and $b(x)$ given functions and $|p| \neq 1$.

$$f(x) = a(x) + b(x)f(px), \quad A = \lim_{i\to\infty} p^i, \quad f(A) \text{ known.} \tag{9.6e}$$

In the region where the following sums and products converge, the solution is given by

$$f(x) = \sum_{j\geq 0} a\left(p^j x\right) \prod_{i=0}^{j-1} b\left(p^i x\right) + f(A) \prod_{i\geq 0} b\left(p^i x\right) \tag{9.6s}$$

$$T(n) = n(\lg n)^{q-1} + 2T(n/2), \quad q \in \mathbb{N} \Rightarrow T(n) \sim (n/q)(\lg n)^q. \tag{9.7}$$

$$\text{Standard quadratic equation: } ax^2 = Bx + x = 0 \Longrightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{9.8}$$