Soundness and Completeness of Sequent Calculus Rules

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Definition: Derivation

Within CSC

1. An axiom Σ is a derivation with conclusion Σ
2. Suppose:
   - d_1 is a derivation with conclusion Σ_1
   - d_2 is a derivation with conclusion Σ_2

   Then the following are derivations with conclusion Σ_3:

   (a) $\dfrac{d_1}{\Sigma_3}$ if $R_1 = \dfrac{\Sigma_1}{\Sigma_3}$ is an instance of a rule of CSC

   (b) $\dfrac{d_1 \qquad d_2}{\Sigma_3}$ if $R_2 = \dfrac{\Sigma_1 \qquad \Sigma_2}{\Sigma_3}$ is an instance of a rule of CSC

Derivations are (certain) trees, inductively generated by 1, 2a, 2b

Induction Principle for Derivations

Within CSC

Suppose that Prop may be true or false of trees of sequents

To prove that Prop(d) for every derivation d, prove:

1. Prop holds for each axiom d
2. Suppose that Prop holds for derivations
   - d_1, a derivation with conclusion Σ_1
   - d_2 is a derivation with conclusion Σ_2

   Then Prop(d) also holds for:

   (a) $\dfrac{d_1}{\Sigma_3}$ if $R_1 = \dfrac{\Sigma_1}{\Sigma_3}$ is an instance of a rule of CSC

   (b) $\dfrac{d_1 \qquad d_2}{\Sigma_3}$ if $R_2 = \dfrac{\Sigma_1 \qquad \Sigma_2}{\Sigma_3}$ is an instance of a rule of CSC

Soundness

A Prop of derivations

A derivation *d* with conclusion Σ of the form $\Gamma \vdash \Delta$ is sound iff

For all models $\mathbb{M}$,
if $\mathbb{M} \models \Gamma$,
then, for some $\delta \in \Delta$,
$\mathbb{M} \models \delta$

Soundness

A Prop of derivations

A derivation d with conclusion Σ of the form $\Gamma \vdash \Delta$ is sound iff

$$\begin{gathered}\text{For all models } \mathbb{M}, \\ \text{if } \mathbb{M} \models \Gamma, \\ \text{then, for some } \delta \in \Delta, \\ \mathbb{M} \models \delta\end{gathered}$$

How can we prove that every derivation d within CSC is sound?

What about Completeness?

Completeness says:

If a sequent Σ is valid,
then Σ is the conclusion of some derivation

Definition: Attempt

Within CSC

1. Σ is an attempt with conclusion Σ iff every formula in Σ is atomic
2. Suppose:
   - d_1 is an attempt with conclusion Σ_1
   - d_2 is an attempt with conclusion Σ_2

   Then the following are attempts with conclusion Σ_3:

   (a) $\dfrac{d_1}{\Sigma_3}$ if $R_1 = \dfrac{\Sigma_1}{\Sigma_3}$ is an instance of a rule of CSC

   (b) $\dfrac{d_1 \qquad d_2}{\Sigma_3}$ if $R_2 = \dfrac{\Sigma_1 \qquad \Sigma_2}{\Sigma_3}$ is an instance of a rule of CSC

Attempts are (certain) trees, inductively generated by 1, 2a, 2b

Attempts: Success and Failure

- An attempt is a *success* iff
  it is a derivation, i.e. each topmost node is an axiom
- It is a *failure* iff
  some topmost node is not an axiom

Attempts Preserve Counterexamples

Suppose that d is an attempt
with conclusion Σ and
some topmost node a non-axiom $\Gamma \vdash \Delta$

For some $\mathbb{M}$ such that $\mathbb{M} \not\models \Gamma \vdash \Delta$,
$\mathbb{M} \not\models \Sigma$

Proof Search

- Build partial derivations bottom-up
- If a topmost sequent $\Gamma \vdash \Delta$
  contains any connectives,
  choose one and apply its rule
- If $\Gamma \vdash \Delta$ has no connectives, it is terminal
  - "Successful" if an axiom
  - "Failing" otherwise

Proof search terminates,
because we reduce connectives in each step

Completeness

Proof search from $\Gamma \vdash \Delta$ yields derivation d

1. $\mathbb{M} \models \Gamma \vdash \Delta$ iff

   for every leaf $\Theta \vdash \Xi$ of d, $\mathbb{M} \models \Theta \vdash \Xi$
2. $\Gamma \vdash \Delta$ is valid iff

   for every leaf $\Theta \vdash \Xi$ of d, $\Theta \vdash \Xi$ is an axiom
3. If proof search from $\Gamma \vdash \Delta$ also yields d'

   d' is a derivation iff d is a derivation

Completeness: Why? I

If $A_1, \ldots, A_j \vdash B_1, \ldots, B_k$ contains only atomic formulas

$A_1, \ldots, A_j \vdash B_1, \ldots, B_k$ is an instance of an axiom

iff

for all models $\mathbb{M}$, $\mathbb{M} \models A_1, \ldots, A_j \vdash B_1, \ldots, B_k$

Completeness: Why? II

Consider

$$R_1 = \frac{\Sigma_1}{\Sigma_2} \qquad \text{or} \qquad R_2 = \frac{\Sigma_0 \qquad \Sigma_1}{\Sigma_2}$$

R_1: $\mathbb{M} \models \Sigma_2$ iff $\mathbb{M} \models \Sigma_1$

R_2: $\mathbb{M} \models \Sigma_2$ iff both $\mathbb{M} \models \Sigma_0$ and $\mathbb{M} \models \Sigma_1$