Quantifier Rules: Soundness and Completeness, 1

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The Rules for Quantifiers

$$\frac{\Gamma \vdash \varphi[t/x], \Delta}{\Gamma \vdash \exists x \,.\, \varphi, \Delta}$$

$$x \notin \mathrm{fv}(\Gamma, \Delta)\ \frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \exists x \,.\, \varphi \vdash \Delta}$$

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \forall x \,.\, \varphi, \Delta}\ x \notin \mathrm{fv}(\Gamma, \Delta)$$

$$\frac{\Gamma, \varphi[t/x] \vdash \Delta}{\Gamma, \forall x \,.\, \varphi \vdash \Delta}$$

Non-free Variables do not Matter

Lemma

If $\eta(x) = \theta(x)$ for all $x \in \mathrm{fv}(t)$, then

$$\mathbb{A}_{\eta}(t) = \mathbb{A}_{\theta}(t)$$

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If $\eta(x) = \theta(x)$ for all $x \in \mathrm{fv}(\varphi)$, then

$$\mathbb{A} \models_{\eta} \varphi \quad \text{iff} \quad \mathbb{A} \models_{\theta} \varphi$$

Non-free Variables do not Matter

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If $\eta(x) = \theta(x)$ for all $x \in \mathrm{fv}(t)$, then

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If $\eta(x) = \theta(x)$ for all $x \in \mathrm{fv}(\varphi)$, then

$$\mathbb{A} \models_{\eta} \varphi \quad \text{iff} \quad \mathbb{A} \models_{\theta} \varphi$$

Corollary: If φ is a sentence (meaning $\mathrm{fv}(\varphi) = \emptyset$), then for all η, θ

$$\mathbb{A} \models_{\eta} \varphi \quad \text{iff} \quad \mathbb{A} \models_{\theta} \varphi$$

Semantics of Capture-avoiding Substitution

Lemma

$$\mathbb{A} \models_\eta \varphi[t/x]$$

iff

$$\mathbb{A} \models_{\eta[x \mapsto \mathbb{A}_\eta(t)]} \varphi$$

whenever

the former is well-defined

Soundness of $\vdash\ \exists$

Rule preserves validity

$$\frac{\Gamma \ \vdash \ \varphi[t/x], \Delta}{\Gamma \ \vdash \ \exists x \ . \ \varphi, \Delta}$$

Suppose: for every model $\mathbb{M}$ and environment η,

if $\mathbb{M} \models_\eta \gamma$ for every $\gamma \in \Gamma$

then either $\mathbb{M} \models_\eta \varphi[t/x]$ or $\mathbb{M} \models_\eta \delta$ for some $\delta \in \Delta$

Then:

if $\mathbb{A} \models_\theta \gamma$ for every $\gamma \in \Gamma$

then either $\mathbb{A} \models_\theta \exists x \ . \ \varphi$ or $\mathbb{A} \models_\theta \delta$ for some $\delta \in \Delta$

Proof approach, 1

if $\mathbb{A} \models_\theta \gamma$ for every $\gamma \in \Gamma$

then either $\mathbb{A} \models_\theta \exists x \,.\, \varphi$ or $\mathbb{A} \models_\theta \delta$ for some $\delta \in \Delta$

Proof approach, 1

$$\begin{array}{ll} \text{if} & \mathbb{A} \models_\theta \gamma \text{ for every } \gamma \in \Gamma \\ \text{then} & \text{either} \quad \mathbb{A} \models_\theta \exists x \,.\, \varphi \quad \text{or} \quad \mathbb{A} \models_\theta \delta \text{ for some } \delta \in \Delta \end{array}$$

enough to show

$$\begin{array}{ll} \text{if} & \mathbb{A} \models_\theta \gamma \text{ for every } \gamma \in \Gamma \\ \text{then} & \text{either} \quad \mathbb{A} \models_{\theta[x \mapsto \mathbb{A}_\eta(t)]} \varphi \quad \text{or} \quad \mathbb{A} \models_\theta \delta \text{ for some } \delta \in \Delta \end{array}$$

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eqv.

$$\begin{array}{ll} \text{if} & \mathbb{A} \models_\theta \gamma \text{ for every } \gamma \in \Gamma \\ \text{then} & \text{either} \quad \mathbb{A} \models_\theta \varphi[t/x] \quad \text{or} \quad \mathbb{A} \models_\theta \delta \text{ for some } \delta \in \Delta \end{array}$$

Proof approach, 2

Want:

if $\mathbb{A} \models_\theta \gamma$ for every $\gamma \in \Gamma$

then either $\mathbb{A} \models_\theta \varphi[t/x]$ or $\mathbb{A} \models_\theta \delta$ for some $\delta \in \Delta$

Assumed:

for every model $\mathbb{M}$ and environment η,

if $\mathbb{M} \models_\eta \gamma$ for every $\gamma \in \Gamma$

then either $\mathbb{M} \models_\eta \varphi[t/x]$ or $\mathbb{M} \models_\eta \delta$ for some $\delta \in \Delta$

Soundness of $\vdash \forall$

Rule preserves validity

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \forall x \,.\, \varphi, \Delta}\; x \notin \mathrm{fv}(\Gamma, \Delta)$$

Suppose: for every model $\mathbb{M}$ and environment η,

if $\mathbb{M} \models_\eta \gamma$ for every $\gamma \in \Gamma$

then either $\mathbb{M} \models_\eta \varphi$ or $\mathbb{M} \models_\eta \delta$ for some $\delta \in \Delta$

Then:

if $\mathbb{A} \models_\theta \gamma$ for every $\gamma \in \Gamma$

then either $\mathbb{A} \models_\theta \forall x \,.\, \varphi$

or $\mathbb{A} \models_\theta \delta$ for some $\delta \in \Delta$

Proof Approach

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$$\begin{array}{rl} \text{if} & \mathbb{A} \models_\theta \gamma \text{ for every } \gamma \in \Gamma \\ \text{then} & \text{either} \quad \mathbb{A} \models_\theta \forall x \,.\, \varphi \\ & \text{or} \qquad \mathbb{A} \models_\theta \delta \text{ for some } \delta \in \Delta \end{array}$$

Proof Approach

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$$\begin{array}{rl} \text{if} & \mathbb{A} \models_\theta \gamma \text{ for every } \gamma \in \Gamma \\ \text{then} & \text{either for all } \theta', \text{ if } \theta' \sim_x \theta, \text{ then} \quad \mathbb{A} \models_{\theta'} \varphi \\ & \text{or} \qquad\qquad \mathbb{A} \models_\theta \delta \text{ for some } \delta \in \Delta \end{array}$$

Proof Approach, 2

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Proof Approach, 2

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since $x \notin \mathrm{fv}(\Gamma, \Delta)$:
for all θ',
if $\theta' \sim_x \theta$, then

$$\begin{array}{rl} \text{if} & \mathbb{A} \models_{\theta'} \gamma \text{ for every } \gamma \in \Gamma \\ \text{then} & \text{either} \quad \mathbb{A} \models_{\theta'} \varphi \\ & \text{or} \qquad\qquad \mathbb{A} \models_{\theta'} \delta \text{ for some } \delta \in \Delta \end{array}$$

Proof Approach, 2

Want:

if $\mathbb{A} \models_\theta \gamma$ for every $\gamma \in \Gamma$

then either for all θ', if $\theta' \sim_x \theta$, then $\mathbb{A} \models_{\theta'} \varphi$

or $\mathbb{A} \models_\theta \delta$ for some $\delta \in \Delta$

since $x \notin \mathrm{fv}(\Gamma, \Delta)$:

for all θ',

if $\theta' \sim_x \theta$, then

if $\mathbb{A} \models_{\theta'} \gamma$ for every $\gamma \in \Gamma$

then either $\mathbb{A} \models_{\theta'} \varphi$

or $\mathbb{A} \models_{\theta'} \delta$ for some $\delta \in \Delta$

Know: for every model $\mathbb{M}$ and environment η,

if $\mathbb{M} \models_\eta \gamma$ for every $\gamma \in \Gamma$

then either $\mathbb{M} \models_\eta \varphi$ or $\mathbb{M} \models_\eta \delta$ for some $\delta \in \Delta$

Soundness

For every derivation d with conclusion $\Gamma \vdash \Delta$, for all $\mathbb{M}$, η:

$$\begin{array}{rll} \text{if} & \mathbb{M} \models_\eta \gamma \text{ for every } \gamma \in \Gamma & \\ \text{then} & \text{either} & \mathbb{M} \models_\eta \varphi \\ & \text{or} & \quad \mathbb{M} \models_\eta \delta \text{ for some } \delta \in \Delta \end{array}$$

(By induction on d as a tree.)