The Duality of Left and Right

Joshua D. Guttman

Worcester Polytechnic Institute

9 February 2010
And vs. Or, Left vs. Right

\[ \Gamma \vdash \varphi \quad \Gamma \vdash \psi \]
\[ \Gamma \vdash \varphi \wedge \psi \]

\[ \Gamma \vdash \varphi \vee \psi \]

\[ \Gamma \vdash \psi \]
\[ \Gamma \vdash \varphi \vee \psi \]

\[ \Gamma, \varphi \vdash \chi \]
\[ \Gamma, \varphi \wedge \psi \vdash \chi \]

\[ \Gamma, \psi \vdash \chi \]
\[ \Gamma, \varphi \vee \psi \vdash \chi \]
Implication: Left vs. Right

\[ \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi} \quad \frac{\Gamma \vdash \varphi \quad \Gamma, \psi \vdash \chi}{\Gamma, \varphi \rightarrow \psi \vdash \chi} \]
Today’s Lecture

- A limitation of the left-right symmetry
- Incompleteness of these rules
  - But: At least they’re sound
- A more symmetric set of rules
  - Proof search
- Completeness of the new rules
Proving $\neg (A \land \neg A)$

\[
\begin{align*}
A, \quad \bot & \vdash \bot & A & \vdash A \\
\hline
A, \quad A \rightarrow \bot & \vdash \bot \\
\hline
A \land \neg A & \vdash \bot \\
\hline
\vdash (A \land \neg A) \rightarrow \bot
\end{align*}
\]
Proving $A \lor \neg A$

\[\frac{\vdash A}{\vdash A \lor \neg A}\]

\[\frac{A \vdash \bot}{\vdash A \rightarrow \bot}\]

\[\frac{\vdash \bot}{\vdash A \lor \neg A}\]

Can’t work with both components $A$, $\neg A$ together on the right of the sequent
These Rules are Incomplete
But at least they’re sound

⊢ $A \lor \neg A$ is not derivable, but
$A \lor \neg A$ is valid: $\emptyset \vdash A \lor \neg A$

- If $\Gamma \vdash \varphi$ is derivable, then $\Gamma \models \varphi$
- How would we prove this?
More Symmetric Sequents

- Instead of
  \[ \Gamma \vdash \varphi \]

  use
  \[ \Gamma \vdash \Delta \]

- Duality of left and right:
  \[ \varphi_1, \ldots, \varphi_j \vdash \psi_1, \ldots, \psi_j \]
  means
  \[ \varphi_1 \land \ldots \land \varphi_j \vdash \psi_1 \lor \ldots \lor \psi_j \]
  Comma means \( \land \) on the left and \( \lor \) on the right
More Symmetric Rules

Sequents $\Gamma \vdash \Delta$

\[
\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \land \psi, \Delta} \quad \frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \land \psi \vdash \Delta}
\]

\[
\frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \lor \psi, \Delta} \quad \frac{\Gamma, \varphi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \lor \psi \vdash \Delta}
\]
Implication: New form

\[\Gamma, \varphi \vdash \psi, \Delta \]

\[\Gamma \vdash \varphi \rightarrow \psi, \Delta \]

\[\Gamma \vdash \varphi, \Delta \quad \Gamma, \psi \vdash \Delta \]

\[\Gamma, \varphi \rightarrow \psi \vdash \Delta \]
Proving $A \lor \neg A$

\[
\frac{A \vdash A, \bot}{\vdash A, A \rightarrow \bot} \frac{\vdash A \lor \neg A}{\vdash A \lor \neg A}
\]
Another Proof

\[
\begin{align*}
A, D & \vdash A, C, \bot & A, C, D & \vdash C, \bot \\
A, A \rightarrow C, D & \vdash C, \bot \\
A \rightarrow C, D & \vdash \neg A, C, \bot \\
A \rightarrow C, D & \vdash \neg A \vee C, \bot \\
A \rightarrow C, D & \vdash \bot, A \rightarrow C, D & \vdash \bot \\
D & \vdash \bot, A \rightarrow C, D & \vdash \bot \\
(\neg A \vee C) & \rightarrow (D \rightarrow \bot), A \rightarrow C, D & \vdash \bot \\
(\neg A \vee C) & \rightarrow (D \rightarrow \bot), A \rightarrow C & \vdash D & \vdash \bot
\end{align*}
\]
Proof Search

- Build partial derivations bottom-up
- If a topmost sequent $\Gamma \vdash \Delta$
  contains any connectives,
  choose one and apply its rule
- If $\Gamma \vdash \Delta$ has no connectives, it is terminal
  - “Successful” if an axiom
  - “Failing” otherwise

Proof search terminates,
because we reduce connectives in each step
Completeness
Proof search from $\Gamma \vdash \Delta$ yields derivation $d$

1. $M \models \Gamma \vdash \Delta$ iff
   for every leaf $\Theta \vdash \Xi$ of $d$, $M \models \Theta \vdash \Xi$

2. $\Gamma \vdash \Delta$ is valid iff
   for every leaf $\Theta \vdash \Xi$ of $d$, $\Theta \vdash \Xi$ is an axiom

3. If proof search from $\Gamma \vdash \Delta$ also yields $d'$
   $d'$ is a derivation iff $d$ is a derivation
Completeness: Why? I

If \( A_1, \ldots, A_j \vdash B_1, \ldots, B_k \) contains only atomic formulas

\[ A_1, \ldots, A_j \vdash B_1, \ldots, B_k \text{ is an instance of an axiom} \]

iff

\[ \text{for all models } M, M \models A_1, \ldots, A_j \vdash B_1, \ldots, B_k \]
Completeness: Why? II

Consider

\[ R_1 = \frac{\Sigma_1}{\Sigma_2} \quad \text{or} \quad R_2 = \frac{\Sigma_0}{\Sigma_2} \frac{\Sigma_1}{\Sigma_2} \]

\[ R_1: \ M \models \Sigma_2 \iff M \models \Sigma_1 \]
\[ R_2: \ M \models \Sigma_2 \iff \text{both } M \models \Sigma_0 \text{ and } M \models \Sigma_1 \]