CS 521, HW 4: Normal Derivations and the Subformula Property
With an Answer to a Corrected (6)

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Non-derivability in Intuitionist Propositional Logic. Use the theorem that every normal derivation has the subformula property to show that the following judgments are not derivable in our system. It is Thm. 23 in the current version of the lecture notes.[1] Say which of these formulas are classically valid, i.e. true for every assignment of truth values to atomic formulas.

$$\vdash\ p \tag{1}$$

$$\vdash\ p \to \bot \tag{2}$$

$$\vdash\ p \vee (p \to \bot) \tag{3}$$

$$\vdash\ ((p \to \bot) \to \bot) \to p \tag{4}$$

Double Negation Elimination has the same strength as Excluded Middle. Formulas of the form given in Eqn. 3 are instances of a law called *the excluded middle*. The idea behind the name is that some middle position between p and $\neg p$ is impossible. Formulas of the form given in Eqn. 4 are instances of a law called *double negation elimination*. The instances of these two rules are equivalent in our logic. Prove:

$$p \vee (p \to \bot)\ \vdash\ ((p \to \bot) \to \bot) \to p \tag{5}$$

$$(((p \vee (p \to \bot)) \to \bot) \to \bot) \to (p \vee (p \to \bot))\ \vdash\ p \vee (p \to \bot) \tag{6}$$

Eqn. 6 has been amended. (:-) It now states that we can derive one instance of the law of excluded middle from an instance of double negation elimination where the doubly negated formula is not p but instead $(p \vee (p \to \bot))$.

[1] At URL `http://web.cs.wpi.edu/~guttman/cs521_website/16sep10_consequence.pdf`.

Answer to Eqn. 6. We can infer the result using implication elimination from the two premises $\phi \to \psi \ \vdash \ \phi \to \psi$ and $\phi \to \psi \ \vdash \ \phi$. Here, ψ is $(p \lor (p \to \bot))$, and ϕ is its double negation.

We skip this step for typographical reasons, and prove the sequent $\vdash \ \phi$, where we omit the (now unnecessary) premise $\phi \to \psi$. First take a few steps:

$$\dfrac{\dfrac{\dfrac{(p \lor (p \to \bot)) \to \bot,\ p \ \vdash \ \bot}{(p \lor (p \to \bot)) \to \bot \ \vdash \ p \to \bot} \qquad (p \lor (p \to \bot)) \to \bot \ \vdash \ p}{(p \lor (p \to \bot)) \to \bot \ \vdash \ \bot}}{\vdash \ ((p \lor (p \to \bot)) \to \bot) \to \bot}$$

Next, we follow the left branch:

$$\dfrac{(p \lor (p \to \bot)) \to \bot,\ p \ \vdash \ (p \lor (p \to \bot)) \to \bot \qquad \dfrac{(p \lor (p \to \bot)) \to \bot,\ p \ \vdash \ p}{(p \lor (p \to \bot)) \to \bot,\ p \ \vdash \ p \lor (p \to \bot)}}{(p \lor (p \to \bot)) \to \bot,\ p \ \vdash \ \bot}$$

Now, follow the right branch:

$$\dfrac{\dfrac{\dfrac{\neg(p \lor \neg p),\ p \ \vdash \ \bot}{\neg(p \lor \neg p) \ \vdash \ \neg p}}{\neg(p \lor \neg p) \ \vdash \ p \lor \neg p} \qquad \neg(p \lor \neg p),\ p \ \vdash \ p \qquad \dfrac{\neg(p \lor \neg p),\ \neg p \ \vdash \ \bot}{\neg(p \lor \neg p),\ \neg p \ \vdash \ p}}{\neg(p \lor \neg p) \ \vdash \ p}$$

Observe that the top left goal was already proved on the left branch above. As for the top right goal:

$$\dfrac{\neg(p \lor \neg p),\ \neg p \ \vdash \ (p \lor \neg p) \to \bot \qquad \dfrac{\neg(p \lor \neg p),\ \neg p \ \vdash \ \neg p}{\neg(p \lor \neg p),\ \neg p \ \vdash \ p \lor \neg p}}{\neg(p \lor \neg p),\ \neg p \ \vdash \ \bot}$$

Proof of Lemma 22. Please choose two clauses of Lemma 22, about paths in normal derivation. For each, choose two relevant inference rules, and show that the assertion is correct for those inference rules.