Horn Clauses and Models for Them
(and a bit about the quiz)

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The Key Point in Quiz

To prove $\text{Prop}(\gamma)$ for all $\gamma$

**Base case** Suppose $\gamma$ is an atom $A \in \mathcal{L}$; then . . .

**Ind. step** Suppose $\text{Prop}(\alpha)$ and $\text{Prop}(\beta)$

Consider formulas

- $\neg \alpha$: . . .
- $\alpha \lor \beta$: . . .
- $\alpha \land \beta$: . . .
Question 1
When $M \leq M'$ and $\gamma$ purely positive, $M \models \gamma$ implies $M' \models \gamma$

Let $M \leq M'$.

**Base case** Suppose $\gamma$ is an atom $A \in \mathcal{L}$.

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Consider formulas

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$\alpha \lor \beta$:

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Question 1

When \( M \leq M' \) and \( \gamma \) purely positive, 
\( M \models \gamma \) implies \( M' \models \gamma \)

Let \( M \leq M' \).

Prop(\( \gamma \)) says:

If \( \gamma \) is purely positive and \( M \models \gamma \), then \( M' \models \gamma \)

Base case  Suppose \( \gamma \) is an atom \( A \in \mathcal{L} \).

Ind. step  Suppose

\[ \text{Prop}(\alpha) \quad \text{and} \quad \text{Prop}(\beta) \]

Consider formulas

\( \neg \alpha : \)
\( \alpha \lor \beta : \)
\( \alpha \land \beta : \)
Question 1

When $\mathcal{M} \leq \mathcal{M}'$ and $\gamma$ purely positive, $\mathcal{M} \models \gamma$ implies $\mathcal{M}' \models \gamma$

Let $\mathcal{M} \leq \mathcal{M}'$.

**Base case** Suppose $\gamma$ is an atom $A \in \mathcal{L}$.

If $\mathcal{M} \models A$, then $\mathcal{M}(A) = 1$.
Thus, $\mathcal{M}'(A) = 1$, and $\mathcal{M}' \models A$

**Ind. step** Suppose $\text{Prop}(\alpha)$ and $\text{Prop}(\beta)$

Consider formulas

$\neg \alpha$:

$\alpha \lor \beta$:

$\alpha \land \beta$:  

Question 1

When $M \leq M'$ and $\gamma$ purely positive, $M \models \gamma$ implies $M' \models \gamma$

Let $M \leq M'$.

**Base case** Suppose $\gamma$ is an atom $A \in \mathcal{L}$.

**Ind. step** Suppose $\text{Prop}(\alpha)$ and $\text{Prop}(\beta)$

Consider formulas

$\neg \alpha$: $\neg \alpha$ is not purely positive

$\alpha \lor \beta$: 

$\alpha \land \beta$: 

Question 1

When $M \leq M'$ and $\gamma$ purely positive, $M \models \gamma$ implies $M' \models \gamma$

Let $M \leq M'$.

Base case Suppose $\gamma$ is an atom $A \in L$.

Ind. step Suppose $\text{Prop}(\alpha)$ and $\text{Prop}(\beta)$

Consider formulas

$\neg \alpha$: $M \models \alpha \lor \beta$ implies $M \models \alpha$ or $M \models \beta$.

So $M' \models \alpha$ or $M' \models \beta$.

Thus $M' \models \alpha \lor \beta$

$\alpha \land \beta$: 
Question 1

When $M \leq M'$ and $\gamma$ purely positive,
$M \models \gamma$ implies $M' \models \gamma$

Let $M \leq M'$.

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**Ind. step** Suppose $\text{Prop}(\alpha)$ and $\text{Prop}(\beta)$

Consider formulas

- $\neg \alpha$
- $\alpha \lor \beta$
- $\alpha \land \beta$:
  $M \models \alpha \land \beta$ implies $M \models \alpha$ and $M \models \beta$.
  So $M' \models \alpha$ and $M' \models \beta$.
  Thus $M' \models \alpha \land \beta$
Question 2
Every formula has an equivalent negation normal form

Base case Suppose \( \gamma \) is an atom \( A \in \mathcal{L} \); then . . .

Ind. step Suppose \( \text{Prop}(\alpha) \) and \( \text{Prop}(\beta) \)

Consider formulas

\[ \neg \alpha: \ldots \]
\[ \alpha \lor \beta: \ldots \]
\[ \alpha \land \beta: \ldots \]
Question 2

Every formula has an equivalent negation normal form

Trick: choose $\text{Prop}(\gamma)$ to be:

both $\gamma$ and $\neg \gamma$ have nnfs

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Ind. step Suppose

$\text{Prop}(\alpha)$ and $\text{Prop}(\beta)$

Consider formulas

$\neg \alpha$:  
$\alpha \lor \beta$:  
$\alpha \land \beta$:  

Horn Clauses

The What

- Literal: Atomic or negated atomic formula $A$ or $\neg A$
- Clause: A disjunction of literals $L_1 \lor \ldots \lor L_k$ or eqv.
  (letting $P_i, Q_i$ be atoms)

\[(P_1 \land \ldots \land P_m) \rightarrow (Q_1 \lor \ldots \lor Q_n)\]

- Horn clause: Clause with 0 or 1 positive literal $n = 0$ or $n = 1$
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$$\neg (P_1 \land \ldots \land P_m) \lor (Q_1 \lor \ldots \lor Q_n)$$

- Horn clause: Clause with 0 or 1 positive literal $n = 0$ or $n = 1$
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  \[(\neg P_1 \lor \ldots \lor \neg P_m) \lor (Q_1 \lor \ldots \lor Q_n)\]
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\[
(P_1 \land \ldots \land P_m) \rightarrow (Q_1 \lor \ldots \lor Q_n)
\]

- Horn clause: Clause with 0 or 1 positive literal \( n = 0 \) or \( n = 1 \)

Some special cases:

- \( m = 1, n = 0 \): \( \neg P_1 \)
- \( n = 0 \): \( \neg(P_1 \land \ldots \land P_m) \)
- \( m = 0, n = 1 \): \( Q_1 \)
- \( n = 1 \): \( (P_1 \land \ldots \land P_m) \rightarrow Q_1 \)
Working with models
Working with models

Their models are closed under taking minima
Models form a lattice, 1

Four models for $\mathcal{L} = \{P, Q\}$

$\langle 1, 1 \rangle$  $\langle 1, 0 \rangle$  $\langle 0, 1 \rangle$  $\langle 0, 0 \rangle$

$\mathbb{M} \leq \mathbb{M}'$ means:

for all $A \in \mathcal{L}$,  $\mathbb{M}(A) \leq \mathbb{M}'(A)$
Models form a lattice, 2

Eight models for $\mathcal{L} = \{P, Q, R\}$
Models form a lattice, 2
Eight models for $\mathcal{L} = \{P, Q, R\}$

Similar, but harder to draw, if $\mathcal{L}$ infinite
Models form a lattice, 2

Eight models for $\mathcal{L} = \{P, Q, R\}$

![Diagram of a lattice with eight models:]

- $\langle 1, 1, 1 \rangle$
- $\langle 0, 1, 1 \rangle$
- $\langle 0, 0, 1 \rangle$
- $\langle 0, 0, 0 \rangle$
- $\langle 1, 0, 1 \rangle$
- $\langle 0, 1, 0 \rangle$
- $\langle 1, 1, 0 \rangle$
- $\langle 1, 0, 0 \rangle$

Complete lattice: Every set of points has a least upper bound and a greatest lower bound
Generic Models

Let $\Sigma$ be a set of sentences and $\mathcal{M} \models \Sigma$

- $\mathcal{M}$ is a generic model for $\Sigma$ iff:
  - For every atom $A \in \mathcal{L}$, $\mathcal{M} \models A$ iff $\Sigma \models A$
  - That is, $\mathcal{M} \models A$ iff every model of $\Sigma$ agrees that $A$
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- Let $\mathcal{M}_0$ be $\inf \{ \mathcal{M} : \mathcal{M} \models \Sigma \}$
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- If $\mathcal{M}_0 \models \Sigma$, then
  
  $\mathcal{M}_0$ is generic for $\Sigma$
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- Otherwise,
  - $\Sigma$ has no generic model
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- Let $\mathcal{M}_0$ be $\inf\{\mathcal{M} : \mathcal{M} \models \Sigma\}$
- If $\mathcal{M}_0 \models \Sigma$, then
  $\mathcal{M}_0$ is generic for $\Sigma$

- Otherwise,
  $\Sigma$ has no generic model
  e.g. $A \lor B$
If $\Sigma$ has a generic model

\[ \Sigma \models P \lor Q \quad \text{implies} \quad \Sigma \models P \text{ or } \Sigma \models Q \]
If $\Sigma$ has a generic model

$$\Sigma \models P \lor Q \quad \text{implies} \quad \Sigma \models P \lor \Sigma \not\models Q$$

Likewise if $\Sigma \models Q_1 \lor \ldots \lor Q_n$
Horn Theories have Generic Models

- $\Sigma$ is Horn,
- $\Sigma$ is satisfiable, and
- $M_0 = \inf \{ M : M \models \Sigma \}$ implies $M_0 \models \Sigma$
Horn Theories have Generic Models

- \( \Sigma \) is Horn,
- \( \Sigma \) is satisfiable, and
- \( \mathcal{M}_0 = \inf \{ \mathcal{M} : \mathcal{M} \models \Sigma \} \)

implies \( \mathcal{M}_0 \models \Sigma \)

Cases:

\( n = 0: \quad \neg(P_1 \wedge \ldots \wedge P_m) \)

\( m = 0, n = 1: \quad Q_1 \)

\( n = 1: \quad (P_1 \wedge \ldots \wedge P_m) \rightarrow Q_1 \)
Horn Theories have Generic Models

- \( \Sigma \) is Horn,
- \( \Sigma \) is satisfiable, and
- \( M_0 = \inf\{M : M \models \Sigma\} \)

implies \( M_0 \models \Sigma \)

Cases:

- \( n = 0 \): \( \neg(P_1 \land \ldots \land P_m) \)
- \( m = 0, n = 1 \): \( Q_1 \)
- \( n = 1 \): \( (P_1 \land \ldots \land P_m) \rightarrow Q_1 \)

But non-Horn theories can have generic models too, e.g.

\[ A \rightarrow (B \lor C) \]
“Stably Generic”

Σ has stably generic models iff:

for every set of atoms $T \subseteq \mathcal{L}$,
if $\Sigma \cup T$ is satisfiable,
then $\Sigma \cup T$ has a generic model
Models form a lattice, 2
Eight models for $\mathcal{L} = \{P, Q, R\}$
“Stably Generic”

Σ has stably generic models iff:

for every set of atoms \( T \subseteq \mathcal{L} \),
if \( \Sigma \cup T \) is satisfiable,
then \( \Sigma \cup T \) has a generic model

\( T \) determines lattice point \( M_T \); if \( \Sigma \cup T \) satisfiable,

\[
\inf \{ M : M_T \leq M \text{ and } M \models \Sigma \}
\]

is also a model of \( \Sigma \cup T \)
Horn Theories $\Sigma$

$\Sigma$ is equivalent to a set $\Sigma'$ of Horn formulas iff $\Sigma$ has stably generic models

$\Sigma'$ is a set of Horn clauses, so has a generic model.

Let $\Sigma' = \{ \alpha : \alpha$ is Horn and $\Sigma \models \alpha \}$.

Let $\Sigma \models \beta$, where $\beta = \left( P_1 \land \ldots \land P_m \right) \rightarrow \left( Q_1 \lor \ldots \lor Q_n \right)$.

So $\Sigma \cup \{ P_1 \land \ldots \land P_m \} \models Q_1 \lor \ldots \lor Q_n$ and $\Sigma \cup \{ P_1 \land \ldots \land P_m \}$ has a generic model.

So $\Sigma \cup \{ P_1 \land \ldots \land P_m \}$ $\models Q_i$ for some $i$ s.t. $1 \leq i \leq n$.
Horn Theories $\Sigma$

$\Sigma$ is equivalent to a set $\Sigma'$ of Horn formulas iff $\Sigma$ has stably generic models

1. right $\Rightarrow$ left:
$\Sigma' \cup T$ is a set of Horn clauses, so has a generic model
Horn Theories $\Sigma$

$\Sigma$ is equivalent to a set $\Sigma'$ of Horn formulas iff $\Sigma$ has stably generic models

1. right $\Rightarrow$ left: $\Sigma' \cup T$ is a set of Horn clauses, so has a generic model

2. left $\Rightarrow$ right: Let $\Sigma' = \{\alpha: \alpha$ is Horn and $\Sigma \models \alpha\}$

Let $\Sigma \models \beta$, where $\beta$ is

$$(P_1 \land \ldots \land P_m) \rightarrow (Q_1 \lor \ldots \lor Q_n)$$

So $\Sigma \cup \{P_1 \land \ldots \land P_m\} \models Q_1 \lor \ldots \lor Q_n$

and $\Sigma \cup \{P_1 \land \ldots \land P_m\}$ has a generic model
If $\Sigma$ has a generic model

$\Sigma \models P \lor Q$ implies $\Sigma \models P$ or $\Sigma \not\models Q$

Likewise if $\Sigma \models Q_1 \lor \ldots \lor Q_n$
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So $\Sigma \cup \{ P_1 \land \ldots \land P_m \} \models Q_1 \lor \ldots \lor Q_n$
and $\Sigma \cup \{ P_1 \land \ldots \land P_m \}$ has a generic model
So $\Sigma \cup \{ P_1 \land \ldots \land P_m \} \models Q_i$ for some $i$ s.t. $1 \leq i \leq n$
Summary

Horn theories defined in terms of syntax
at most one positive literal per clause
but characterize property of models
stably generic models,
i.e. even after adding atoms,
generic if satisfiable