Horn Clauses and Models for Them

(and a bit about the quiz)

Joshua D. Guttman

Worcester Polytechnic Institute

28 January 2010

The Key Point in Quiz

To prove Prop(γ) for all γ

Base case Suppose γ is an atom $A \in \mathcal{L}$; then ...

Ind. step Suppose

$$\text{Prop}(\alpha) \quad \text{and} \quad \text{Prop}(\beta)$$

Consider formulas

$\neg\alpha$: ...

$\alpha \vee \beta$: ...

$\alpha \wedge \beta$: ...

Question 1

When $\mathbb{M} \leq \mathbb{M}'$ and γ purely positive,
$\mathbb{M} \models \gamma$ implies $\mathbb{M}' \models \gamma$

Let $\mathbb{M} \leq \mathbb{M}'$.

Base case Suppose γ is an atom $A \in \mathcal{L}$.

Ind. step Suppose

$$\text{Prop}(\alpha) \quad \text{and} \quad \text{Prop}(\beta)$$

Consider formulas

$\neg\alpha$:
$\alpha \vee \beta$:
$\alpha \wedge \beta$:

Question 1

When $\mathbb{M} \leq \mathbb{M}'$ and γ purely positive,
$\mathbb{M} \models \gamma$ implies $\mathbb{M}' \models \gamma$

Let $\mathbb{M} \leq \mathbb{M}'$.

Prop(γ) says:

If γ is purely positive and $\mathbb{M} \models \gamma$, then $\mathbb{M}' \models \gamma$

Base case Suppose γ is an atom $A \in \mathcal{L}$.

Ind. step Suppose

$$\text{Prop}(\alpha) \quad \text{and} \quad \text{Prop}(\beta)$$

Consider formulas

$\neg\alpha$:
$\alpha \vee \beta$:
$\alpha \wedge \beta$:

Question 1

When $\mathbb{M} \leq \mathbb{M}'$ and γ purely positive,
$\mathbb{M} \models \gamma$ implies $\mathbb{M}' \models \gamma$

Let $\mathbb{M} \leq \mathbb{M}'$.

Base case Suppose γ is an atom $A \in \mathcal{L}$.
If $\mathbb{M} \models A$, then $\mathbb{M}(A) = 1$.
Thus, $\mathbb{M}'(A) = 1$, and $\mathbb{M}' \models A$

Ind. step Suppose

$$\text{Prop}(\alpha) \quad \text{and} \quad \text{Prop}(\beta)$$

Consider formulas

$\neg\alpha$:
$\alpha \vee \beta$:
$\alpha \wedge \beta$:

Question 1

When $\mathbb{M} \leq \mathbb{M}'$ and γ purely positive,
$\mathbb{M} \models \gamma$ implies $\mathbb{M}' \models \gamma$

Let $\mathbb{M} \leq \mathbb{M}'$.

Base case Suppose γ is an atom $A \in \mathcal{L}$.

Ind. step Suppose

$$\text{Prop}(\alpha) \quad \text{and} \quad \text{Prop}(\beta)$$

Consider formulas

$\neg\alpha$: $\neg\alpha$ is not purely positive

$\alpha \vee \beta$:

$\alpha \wedge \beta$:

Question 1

When $\mathbb{M} \leq \mathbb{M}'$ and γ purely positive,
$\mathbb{M} \models \gamma$ implies $\mathbb{M}' \models \gamma$

Let $\mathbb{M} \leq \mathbb{M}'$.

Base case Suppose γ is an atom $A \in \mathcal{L}$.

Ind. step Suppose

$$\text{Prop}(\alpha) \quad \text{and} \quad \text{Prop}(\beta)$$

Consider formulas

$\neg\alpha$:

$\alpha \vee \beta$: $\mathbb{M} \models \alpha \vee \beta$ implies
$\mathbb{M} \models \alpha$ or $\mathbb{M} \models \beta$.
So $\mathbb{M}' \models \alpha$ or $\mathbb{M}' \models \beta$.
Thus $\mathbb{M}' \models \alpha \vee \beta$

$\alpha \wedge \beta$:

Question 1

When $\mathbb{M} \leq \mathbb{M}'$ and γ purely positive,
$\mathbb{M} \models \gamma$ implies $\mathbb{M}' \models \gamma$

Let $\mathbb{M} \leq \mathbb{M}'$.

Base case Suppose γ is an atom $A \in \mathcal{L}$.

Ind. step Suppose

$$\text{Prop}(\alpha) \quad \text{and} \quad \text{Prop}(\beta)$$

Consider formulas

$\neg\alpha$:

$\alpha \vee \beta$:

$\alpha \wedge \beta$: $\mathbb{M} \models \alpha \wedge \beta$ implies
$\mathbb{M} \models \alpha$ and $\mathbb{M} \models \beta$.
So $\mathbb{M}' \models \alpha$ and $\mathbb{M}' \models \beta$.
Thus $\mathbb{M}' \models \alpha \wedge \beta$

Question 2

Every formula has an equivalent negation normal form

Base case Suppose γ is an atom $A \in \mathcal{L}$; then ...

Ind. step Suppose

$$\text{Prop}(\alpha) \quad \text{and} \quad \text{Prop}(\beta)$$

Consider formulas

$\neg\alpha$: ...

$\alpha \vee \beta$: ...

$\alpha \wedge \beta$: ...

Question 2

Every formula has an equivalent negation normal form

Trick: choose Prop(γ) to be:

$$\text{both } \gamma \text{ and } \neg\gamma \text{ have nnfs}$$

Base case Suppose γ is an atom $A \in \mathcal{L}$; then ...

Ind. step Suppose

$$\text{Prop}(\alpha) \quad \text{and} \quad \text{Prop}(\beta)$$

Consider formulas

$\neg\alpha$: ...

$\alpha \vee \beta$: ...

$\alpha \wedge \beta$: ...

Horn Clauses

The What

- Literal: Atomic or negated atomic formula A or $\neg A$
- Clause: A disjunction of literals $L_1 \vee \ldots \vee L_k$ or eqv. (letting P_i, Q_i be atoms)

$$(P_1 \wedge \ldots \wedge P_m) \rightarrow (Q_1 \vee \ldots \vee Q_n)$$

- Horn clause: Clause with 0 or 1 positive literal $n = 0$ or $n = 1$

Horn Clauses

The What

- Literal: Atomic or negated atomic formula A or $\neg A$
- Clause: A disjunction of literals $L_1 \vee \ldots \vee L_k$ or eqv. (letting P_i, Q_i be atoms)

$$\neg(P_1 \wedge \ldots \wedge P_m) \vee (Q_1 \vee \ldots \vee Q_n)$$

- Horn clause: Clause with 0 or 1 positive literal $n = 0$ or $n = 1$

Horn Clauses

The What

- Literal: Atomic or negated atomic formula A or $\neg A$
- Clause: A disjunction of literals $L_1 \vee \ldots \vee L_k$ or eqv. (letting P_i, Q_i be atoms)

$$(\neg P_1 \vee \ldots \vee \neg P_m) \vee (Q_1 \vee \ldots \vee Q_n)$$

- Horn clause: Clause with 0 or 1 positive literal $n = 0$ or $n = 1$

Horn Clauses

The What

- Literal: Atomic or negated atomic formula A or $\neg A$
- Clause: A disjunction of literals $L_1 \vee \ldots \vee L_k$ or eqv. (letting P_i, Q_i be atoms)

$$(P_1 \wedge \ldots \wedge P_m) \rightarrow (Q_1 \vee \ldots \vee Q_n)$$

- Horn clause: Clause with 0 or 1 positive literal $n = 0$ or $n = 1$

Horn Clauses

The What

- Literal: Atomic or negated atomic formula A or $\neg A$
- Clause: A disjunction of literals $L_1 \vee \ldots \vee L_k$ or eqv. (letting P_i, Q_i be atoms)

$$(P_1 \wedge \ldots \wedge P_m) \rightarrow (Q_1 \vee \ldots \vee Q_n)$$

- Horn clause: Clause with 0 or 1 positive literal $n = 0$ or $n = 1$

Some special cases:

$$\begin{array}{rl} m = 1, n = 0: & \neg P_1 \\ n = 0: & \neg(P_1 \wedge \ldots \wedge P_m) \\ m = 0, n = 1: & Q_1 \\ n = 1: & (P_1 \wedge \ldots \wedge P_m) \rightarrow Q_1 \end{array}$$

Working with models

Horn Clauses

The Why

Working with models

Their models are closed under taking minima

Models form a lattice, 1

Four models for $\mathcal{L} = \{P, Q\}$

$$\begin{tikzcd}
 & \langle 1,1 \rangle & \\
\langle 0,1 \rangle \arrow[ur] & & \langle 1,0 \rangle \arrow[ul] \\
 & \langle 0,0 \rangle \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$\mathbb{M} \leq \mathbb{M}'$ means:

for all $A \in \mathcal{L}$, $\quad \mathbb{M}(A) \leq \mathbb{M}'(A)$

Models form a lattice, 2

Eight models for $\mathcal{L} = \{P, Q, R\}$

$$
\begin{tikzcd}
 & & \langle 1,1,1 \rangle & & \\
\langle 0,1,1 \rangle \arrow[urr] & & \langle 1,0,1 \rangle \arrow[u] & & \langle 1,1,0 \rangle \arrow[ull] \\
\langle 0,0,1 \rangle \arrow[u] \arrow[urr] & & \langle 0,1,0 \rangle \arrow[ull] \arrow[urr] & & \langle 1,0,0 \rangle \arrow[ull] \arrow[u] \\
 & & \langle 0,0,0 \rangle \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}
$$

Models form a lattice, 2

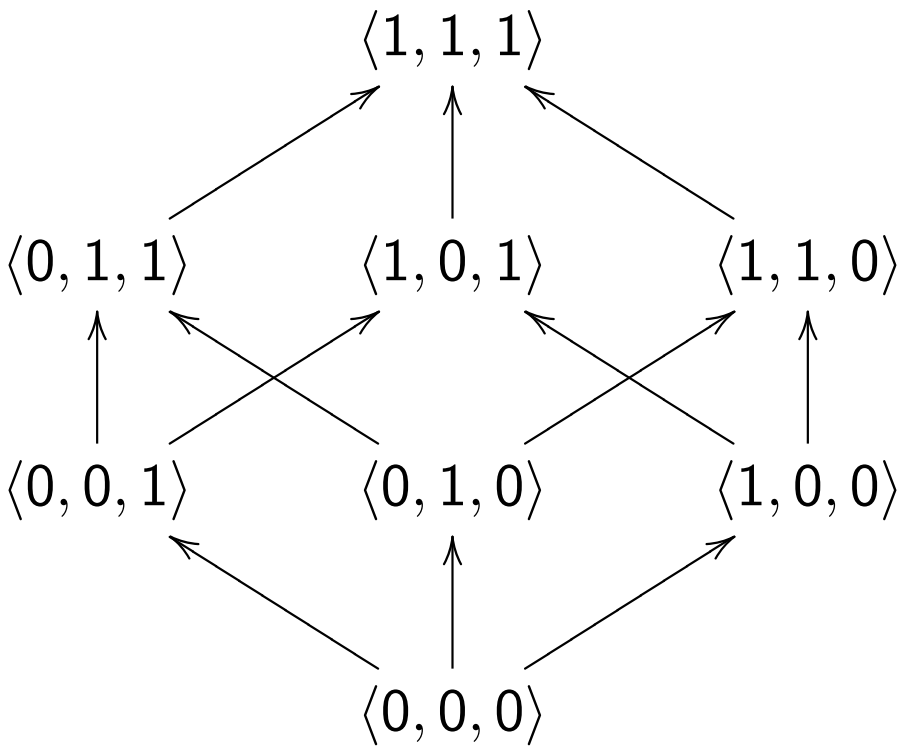
Eight models for $\mathcal{L} = \{P, Q, R\}$

$$
\begin{tikzcd}
 & & \langle 1,1,1\rangle & & \\
\langle 0,1,1\rangle \arrow[urr] & & \langle 1,0,1\rangle \arrow[u] & & \langle 1,1,0\rangle \arrow[ull] \\
\langle 0,0,1\rangle \arrow[u] \arrow[urr] & & \langle 0,1,0\rangle \arrow[ull] \arrow[urr] & & \langle 1,0,0\rangle \arrow[ull] \arrow[u] \\
 & & \langle 0,0,0\rangle \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}
$$

Similar, but harder to draw, if $\mathcal{L}$ infinite

Models form a lattice, 2

Eight models for $\mathcal{L} = \{P, Q, R\}$



Complete lattice: Every set of points has
a least upper bound and a greatest lower bound

Generic Models

Let Σ be a set of sentences and $\mathcal{M} \models \Sigma$

- $\mathcal{M}$ is a generic model for Σ iff:

  For every atom $A \in \mathcal{L}$, $\mathcal{M} \models A$ iff $\Sigma \Vdash A$

  That is,

  $\mathcal{M} \models A$ iff every model of Σ agrees that A

Generic Models

Let Σ be a set of sentences and $\mathcal{M} \models \Sigma$

- $\mathcal{M}$ is a generic model for Σ iff:

  For every atom $A \in \mathcal{L}$, $\mathcal{M} \models A$ iff $\Sigma \Vdash A$

  That is,

  $\mathcal{M} \models A$ iff every model of Σ agrees that A
- Let $\mathcal{M}_0$ be $\inf\{\mathcal{M} : \mathcal{M} \models \Sigma\}$

Generic Models

Let Σ be a set of sentences and $\mathcal{M} \models \Sigma$

- $\mathcal{M}$ is a generic model for Σ iff:

  For every atom $A \in \mathcal{L}$, $\mathcal{M} \models A$ iff $\Sigma \Vdash A$

  That is,

  $\mathcal{M} \models A$ iff every model of Σ agrees that A
- Let $\mathcal{M}_0$ be $\inf\{\mathcal{M} : \mathcal{M} \models \Sigma\}$
- If $\mathcal{M}_0 \models \Sigma$, then

  $\mathcal{M}_0$ is generic for Σ

Generic Models

Let Σ be a set of sentences and $\mathcal{M} \models \Sigma$

- $\mathcal{M}$ is a generic model for Σ iff:

  For every atom $A \in \mathcal{L}$, $\mathcal{M} \models A$ iff $\Sigma \Vdash A$

  That is,

  $\mathcal{M} \models A$ iff every model of Σ agrees that A
- Let $\mathcal{M}_0$ be $\inf\{\mathcal{M} : \mathcal{M} \models \Sigma\}$
- If $\mathcal{M}_0 \models \Sigma$, then

  $\mathcal{M}_0$ is generic for Σ
- Otherwise,

  Σ has no generic model

Generic Models

Let Σ be a set of sentences and $\mathcal{M} \models \Sigma$

- $\mathcal{M}$ is a generic model for Σ iff:

  For every atom $A \in \mathcal{L}$, $\mathcal{M} \models A$ iff $\Sigma \Vdash A$

  That is,

  $\mathcal{M} \models A$ iff every model of Σ agrees that A
- Let $\mathcal{M}_0$ be $\inf\{\mathcal{M} : \mathcal{M} \models \Sigma\}$
- If $\mathcal{M}_0 \models \Sigma$, then

  $\mathcal{M}_0$ is generic for Σ
- Otherwise,

  Σ has no generic model

  e.g. $A \vee B$

If Σ has a generic model

$$\Sigma \Vdash P \vee Q \quad \text{implies} \quad \Sigma \Vdash P \text{ or } \Sigma \Vdash Q$$

$$\begin{tikzcd}
 & & \langle 1,1,1\rangle & & \\
\langle 0,1,1\rangle \arrow[urr] & & \langle 1,0,1\rangle \arrow[u] & & \langle 1,1,0\rangle \arrow[ull] \\
\langle 0,0,1\rangle \arrow[u] \arrow[urr] & & \langle 0,1,0\rangle \arrow[ull] \arrow[urr] & & \langle 1,0,0\rangle \arrow[ull] \arrow[u] \\
 & & \langle 0,0,0\rangle \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

If Σ has a generic model

$$\Sigma \Vdash P \vee Q \quad \text{implies} \quad \Sigma \Vdash P \text{ or } \Sigma \Vdash Q$$

$$\begin{tikzcd}
 & & \langle 1,1,1\rangle & & \\
\langle 0,1,1\rangle \arrow[urr] & & \langle 1,0,1\rangle \arrow[u] & & \langle 1,1,0\rangle \arrow[ull] \\
\langle 0,0,1\rangle \arrow[u] \arrow[urr] & & \langle 0,1,0\rangle \arrow[ull] \arrow[urr] & & \langle 1,0,0\rangle \arrow[ull] \arrow[u] \\
 & & \langle 0,0,0\rangle \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

Likewise if $\Sigma \Vdash Q_1 \vee \ldots \vee Q_n$

Horn Theories have Generic Models

- Σ is Horn,
- Σ is satisfiable, and
- $\mathcal{M}_0 = \inf\{\mathcal{M} : \mathcal{M} \models \Sigma\}$

implies $\mathcal{M}_0 \models \Sigma$

Horn Theories have Generic Models

- Σ is Horn,
- Σ is satisfiable, and
- $\mathcal{M}_0 = \inf\{\mathcal{M} : \mathcal{M} \models \Sigma\}$

implies $\mathcal{M}_0 \models \Sigma$

Cases:

$$\begin{array}{rl} n = 0: & \neg(P_1 \wedge \ldots \wedge P_m) \\ m = 0, n = 1: & Q_1 \\ n = 1: & (P_1 \wedge \ldots \wedge P_m) \rightarrow Q_1 \end{array}$$

Horn Theories have Generic Models

- Σ is Horn,
- Σ is satisfiable, and
- $\mathcal{M}_0 = \inf\{\mathcal{M} : \mathcal{M} \models \Sigma\}$

implies $\mathcal{M}_0 \models \Sigma$

Cases:

$$\begin{array}{rl} n = 0: & \neg(P_1 \wedge \ldots \wedge P_m) \\ m = 0, n = 1: & Q_1 \\ n = 1: & (P_1 \wedge \ldots \wedge P_m) \rightarrow Q_1 \end{array}$$

But non-Horn theories can have generic models too, e.g.

$$A \rightarrow (B \vee C)$$

"Stably Generic"

Σ has stably generic models iff:

for every set of atoms $T \subseteq \mathcal{L}$,
if $\Sigma \cup T$ is satisfiable,
then $\Sigma \cup T$ has a generic model

Models form a lattice, 2

Eight models for $\mathcal{L} = \{P, Q, R\}$

$$
\begin{tikzcd}
 & & \langle 1,1,1 \rangle & & \\
\langle 0,1,1 \rangle \arrow[urr] & & \langle 1,0,1 \rangle \arrow[u] & & \langle 1,1,0 \rangle \arrow[ull] \\
\langle 0,0,1 \rangle \arrow[u] \arrow[urr] & & \langle 0,1,0 \rangle \arrow[ull] \arrow[urr] & & \langle 1,0,0 \rangle \arrow[ull] \arrow[u] \\
 & & \langle 0,0,0 \rangle \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}
$$

"Stably Generic"

Σ has stably generic models iff:

for every set of atoms $T \subseteq \mathcal{L}$,
if $\Sigma \cup T$ is satisfiable,
then $\Sigma \cup T$ has a generic model

T determines lattice point $\mathbb{M}_T$; if $\Sigma \cup T$ satisfiable,

$$\inf\{\mathbb{M} \colon \mathbb{M}_T \leq \mathbb{M} \text{ and } \mathbb{M} \models \Sigma\}$$

is also a model of $\Sigma \cup T$

Horn Theories Σ

Σ is equivalent to a set Σ' of Horn formulas iff
Σ has stably generic models

Horn Theories Σ

Σ is equivalent to a set Σ' of Horn formulas iff
Σ has stably generic models

1. right $\Rightarrow$ left:
   $\Sigma' \cup T$ is a set of Horn clauses, so has a generic model

Horn Theories Σ

Σ is equivalent to a set Σ' of Horn formulas iff
Σ has stably generic models

1. right $\Rightarrow$ left:
   $\Sigma' \cup T$ is a set of Horn clauses, so has a generic model
2. left $\Rightarrow$ right: Let $\Sigma' =$

$$\{\alpha \colon \alpha \text{ is Horn and } \Sigma \Vdash \alpha\}$$

Let $\Sigma \Vdash \beta$, where β is

$$(P_1 \wedge \ldots \wedge P_m) \rightarrow (Q_1 \vee \ldots \vee Q_n)$$

So $\Sigma \cup \{P_1 \wedge \ldots \wedge P_m\} \Vdash Q_1 \vee \ldots \vee Q_n$
and $\Sigma \cup \{P_1 \wedge \ldots \wedge P_m\}$ has a generic model

If Σ has a generic model

$$\Sigma \Vdash P \vee Q \quad \text{implies} \quad \Sigma \Vdash P \text{ or } \Sigma \Vdash Q$$

$$\begin{tikzcd}
 & & \langle 1,1,1\rangle & & \\
\langle 0,1,1\rangle \arrow[urr] & & \langle 1,0,1\rangle \arrow[u] & & \langle 1,1,0\rangle \arrow[ull] \\
\langle 0,0,1\rangle \arrow[u] \arrow[urr] & & \langle 0,1,0\rangle \arrow[ull] \arrow[urr] & & \langle 1,0,0\rangle \arrow[ull] \arrow[u] \\
 & & \langle 0,0,0\rangle \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

Likewise if $\Sigma \Vdash Q_1 \vee \ldots \vee Q_n$

Horn Theories Σ

Σ is equivalent to a set Σ' of Horn formulas iff
Σ has stably generic models

1. right $\Rightarrow$ left:
   $\Sigma' \cup T$ is a set of Horn clauses, so has a generic model
2. left $\Rightarrow$ right: Let $\Sigma' =$

$$\{\alpha \colon \alpha \text{ is Horn and } \Sigma \Vdash \alpha\}$$

Let $\Sigma \Vdash \beta$, where β is

$$(P_1 \wedge \ldots \wedge P_m) \rightarrow (Q_1 \vee \ldots \vee Q_n)$$

So $\Sigma \cup \{P_1 \wedge \ldots \wedge P_m\} \Vdash Q_1 \vee \ldots \vee Q_n$
and $\Sigma \cup \{P_1 \wedge \ldots \wedge P_m\}$ has a generic model
So $\Sigma \cup \{P_1 \wedge \ldots \wedge P_m\} \Vdash Q_i \quad$ for some i s.t. $1 \leq i \leq n$

Summary

Horn theories defined in terms of syntax

at most one positive literal per clause

but characterize property of models

stably generic models,
i.e. even after adding atoms,
generic if satisfiable