Horn Clauses and Models for Them
(and a bit about the quiz)

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The Key Point in Quiz
To prove Prop(γ) for all γ

Base case Suppose γ is an atom A ∈ L; then . . .
Ind. step Suppose Prop(α) and Prop(β)
Consider formulas
¬α: . . .
α ∨ β: . . .
α ∧ β: . . .

Question 1
When M ≤ M′ and γ purely positive,
M |= γ implies M′ |= γ
Let M ≤ M′.
Prop(γ) says:
If γ is purely positive and M |= γ, then M′ |= γ

Base case Suppose γ is an atom A ∈ L.
Ind. step Suppose Prop(α) and Prop(β)
Consider formulas
¬α: ¬α is not purely positive
α ∨ β: . . .
α ∧ β: . . .
Question 1
When $M \leq M'$ and $\gamma$ purely positive, $M \models \gamma$ implies $M' \models \gamma$

Let $M \leq M'$.

Base case
Suppose $\gamma$ is an atom $A \in \mathcal{L}$.

Ind. step
Suppose $\alpha \lor \beta$ and $\alpha \land \beta$.

Consider formulas

\[ \neg \alpha: \quad \ldots \]
\[ \alpha \lor \beta: \quad \ldots \]
\[ \alpha \land \beta: \quad \ldots \]

Base case
Suppose $\gamma$ is an atom $A \in \mathcal{L}$.

Ind. step
Suppose $\alpha \lor \beta$ and $\alpha \land \beta$.

Consider formulas

\[ \neg \alpha: \quad \ldots \]
\[ \alpha \lor \beta: \quad \ldots \]
\[ \alpha \land \beta: \quad \ldots \]

Question 2
Every formula has an equivalent negation normal form

Trick: choose Prop($\gamma$) to be:

\[ \text{both } \gamma \text{ and } \neg \gamma \text{ have nnfs} \]

Base case
Suppose $\gamma$ is an atom $A \in \mathcal{L}$; then ... 

Ind. step
Suppose $\alpha \lor \beta$ and $\alpha \land \beta$.

Consider formulas

\[ \neg \alpha: \quad \ldots \]
\[ \alpha \lor \beta: \quad \ldots \]
\[ \alpha \land \beta: \quad \ldots \]

Horn Clauses
The What

- Literal: Atomic or negated atomic formula $A$ or $\neg A$
- Clause: A disjunction of literals $L_1 \lor \ldots \lor L_k$ or equiv.
  (letting $P_i, Q_i$ be atoms)

\[ (P_1 \land \ldots \land P_m) \rightarrow (Q_1 \lor \ldots \lor Q_n) \]

- Horn clause: Clause with 0 or 1 positive literal $n = 0$ or $n = 1$
Horn Clauses

The What

- Literal: Atomic or negated atomic formula \( A \) or \( \neg A \)
- Clause: A disjunction of literals \( L_1 \lor \ldots \lor L_k \) or eqv. (letting \( P_i, Q_j \) be atoms)

\[
(\neg P_1 \lor \ldots \lor \neg P_m) \lor (Q_1 \lor \ldots \lor Q_n)
\]

- Horn clause: Clause with 0 or 1 positive literal \( n = 0 \) or \( n = 1 \)

Horn Clauses

The Why

- Literal: Atomic or negated atomic formula \( A \) or \( \neg A \)
- Clause: A disjunction of literals \( L_1 \lor \ldots \lor L_k \) or eqv. (letting \( P_i, Q_j \) be atoms)

\[
(P_1 \land \ldots \land P_m) \rightarrow (Q_1 \lor \ldots \lor Q_n)
\]

- Horn clause: Clause with 0 or 1 positive literal \( n = 0 \) or \( n = 1 \)

Some special cases:
- \( m = 1, n = 0 \): \( \neg P_1 \)
- \( n = 0 \): \( \neg (P_1 \land \ldots \land P_m) \)
- \( m = 0, n = 1 \): \( Q_1 \)
- \( n = 1 \): \( (P_1 \land \ldots \land P_m) \rightarrow Q_1 \)

Models form a lattice, 1

Four models for \( \mathcal{L} = \{P, Q\} \):

\[
\begin{align*}
\langle 1, 1 \rangle & \rightarrow \langle 1, 0 \rangle \\
\langle 0, 1 \rangle & \rightarrow \langle 0, 0 \rangle
\end{align*}
\]

\( M \leq M' \) means:

for all \( A \in \mathcal{L} \), \( M(A) \leq M'(A) \)
Models form a lattice, 2
Eight models for $L = \{P, Q, R\}$

\[
\begin{array}{ccc}
(1,1,1) & (0,1,1) & (1,1,0) \\
(0,1,0) & (1,0,0) & (0,0,0) \\
\end{array}
\]

Complete lattice: Every set of points has a least upper bound and a greatest lower bound

Generic Models
Let $\Sigma$ be a set of sentences and $M \models \Sigma$.

- $M$ is a generic model for $\Sigma$ iff:
  - For every atom $A \in L$, $M \models A$ iff $\Sigma \models A$
  - That is, $M \models A$ iff every model of $\Sigma$ agrees that $A$
  - Let $M_0$ be $\inf \{ M : M \models \Sigma \}$
  - If $M_0 \models \Sigma$, then $M_0$ is generic for $\Sigma$

Similar, but harder to draw, if $L$ infinite
Generic Models

Let \( \Sigma \) be a set of sentences and \( \mathcal{M} \models \Sigma \)

- \( \mathcal{M} \) is a generic model for \( \Sigma \) iff:
  - For every atom \( A \in \mathcal{L} \), \( \mathcal{M} \models A \) iff \( \Sigma \vdash A \)

That is,
- \( \mathcal{M} \models A \) iff every model of \( \Sigma \) agrees that \( A \) is true.

Let \( \mathcal{M}_0 \) be inf\( \{ \mathcal{M} : \mathcal{M} \models \Sigma \} \)

- If \( \mathcal{M}_0 \models \Sigma \), then
  - \( \mathcal{M}_0 \) is generic for \( \Sigma \)
- Otherwise,
  - \( \Sigma \) has no generic model

If \( \Sigma \) has a generic model

\[ \Sigma \models P \lor Q \implies \Sigma \models P \lor Q \]

Likewise if \( \Sigma \models Q_1 \lor \ldots \lor Q_n \)

Horn Theories have Generic Models

- \( \Sigma \) is Horn,
- \( \Sigma \) is satisfiable, and
- \( \mathcal{M}_0 = \inf\{ \mathcal{M} : \mathcal{M} \models \Sigma \} \)

implies \( \mathcal{M}_0 \models \Sigma \)

Cases:
- \( n = 0 \):
  - \( \neg (P_1 \land \ldots \land P_m) \)
- \( m = 0 \), \( n = 1 \):
  - \( Q_1 \)
- \( n = 1 \):
  - \( (P_1 \land \ldots \land P_m) \rightarrow Q_1 \)
Horn Clauses

Horn Theories have Generic Models

- $\Sigma$ is Horn,
- $\Sigma$ is satisfiable, and
- $M_0 = \inf \{ M : M \models \Sigma \}$
  implies $M_0 \models \Sigma$

Cases:

$n = 0$: $(P_1 \land \ldots \land P_m)$

$m = 0, n = 1$: $Q_1$

$n = 1$: $(P_1 \land \ldots \land P_m) \rightarrow Q_1$

But non-Horn theories can have generic models too, e.g.

$A \rightarrow (B \lor C)$

Models form a lattice, 2

Eight models for $L = \{ P, Q, R \}$

```
(1, 1, 1)
(0, 1, 1)   (1, 0, 1)   (1, 1, 0)
(0, 0, 1)   (0, 1, 0)   (1, 0, 0)
(0, 0, 0)
```

"Stably Generic"

$\Sigma$ has stably generic models iff:

- for every set of atoms $T \subseteq L$,
  if $\Sigma \cup T$ is satisfiable,
  then $\Sigma \cup T$ has a generic model

T determines lattice point $M_T$; if $\Sigma \cup T$ satisfiable,

$\inf \{ M : M \leq M_T \text{ and } M \models \Sigma \}$

is also a model of $\Sigma \cup T$

Horn Theories $\Sigma$

$\Sigma$ is equivalent to a set $\Sigma'$ of Horn formulas iff

$\Sigma$ has stably generic models

\begin{itemize}
  \item right $\Rightarrow$ left:
    $\Sigma' \cup T$ is a set of Horn clauses, so has a generic model
\end{itemize}
Horn Theories \( \Sigma \)

\( \Sigma \) is equivalent to a set \( \Sigma' \) of Horn formulas iff

- right \( \Rightarrow \) left:
  \( \Sigma' \cup T \) is a set of Horn clauses, so has a generic model

- left \( \Rightarrow \) right: Let \( \Sigma' = \{ \alpha : \alpha \text{ is Horn and } \Sigma \vdash \alpha \} \)

Let \( \Sigma \Vdash \beta \), where \( \beta \) is

\[(P_1 \land \ldots \land P_m) \rightarrow (Q_1 \lor \ldots \lor Q_n)\]

So \( \Sigma \cup \{P_1 \land \ldots \land P_m\} \Vdash Q_1 \lor \ldots \lor Q_n \)

and \( \Sigma \cup \{P_1 \land \ldots \land P_m\} \) has a generic model

So \( \Sigma \cup \{P_1 \land \ldots \land P_m\} \Vdash Q_i \) for some \( i \) s.t. \( 1 \leq i \leq n \)

If \( \Sigma \) has a generic model

\( \Sigma \Vdash P \lor Q \) implies \( \Sigma \Vdash P \) or \( \Sigma \Vdash Q \)

Likewise if \( \Sigma \Vdash Q_1 \lor \ldots \lor Q_n \)

Summary

Horn theories defined in terms of syntax

- at most one positive literal per clause
- but characterize property of models
- stably generic models,
  i.e. even after adding atoms,
  generic if satisfiable