Natural Deduction: Making Proofs Explicit

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Today’s Goal

Proofs are computational objects. We want to show their computational content explicitly in our formal system.
Conjunction Rules

\[
\begin{align*}
\Gamma \vdash \varphi & \quad \Gamma \vdash \psi \\
\hline
\Gamma \vdash \varphi \land \psi
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash \varphi \land \psi \\
\hline
\Gamma \vdash \varphi
\end{align*}
\]
Conjunction Rules

\[ \Gamma \vdash s: \varphi \quad \Gamma \vdash t: \psi \]
\[ \Gamma \vdash \langle s, t \rangle: \varphi \land \psi \]

\[ \Gamma \vdash \varphi \land \psi \]
\[ \Gamma \vdash \varphi \]

\[ \Gamma \vdash \varphi \land \psi \]
\[ \Gamma \vdash \psi \]
Conjunction Rules

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\[ \Gamma \vdash \langle s, t \rangle : \varphi \land \psi \]

\[ \Gamma \vdash s : \varphi \land \psi \]
\[ \Gamma \vdash \text{fst}(s) : \varphi \]

\[ \Gamma \vdash \varphi \land \psi \]
\[ \Gamma \vdash \psi \]
Conjunction Rules

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\Gamma \vdash s : \varphi \quad \Gamma \vdash t : \psi \\
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\Gamma \vdash s : \varphi \land \psi \\
\Gamma \vdash \text{fst}(s) : \varphi
\]

\[
\Gamma \vdash s : \varphi \land \psi \\
\Gamma \vdash \text{scd}(s) : \psi
\]
Axiom and Bottom

\[\Gamma, \varphi \vdash \varphi\]

\[\Gamma \vdash \bot\]
Axiom and Bottom

\[ \Gamma, x : \varphi \vdash x : \varphi \]

\[ \Gamma \vdash \bot \]

\[ \Gamma \vdash \varphi \]
A Derivation

\[ p \land q \vdash p \land q \]
\[ p \land q \vdash p \]
\[ \vdash (p \land q) \rightarrow p \]
A Derivation

\[
\begin{align*}
&\quad x : p \land q \vdash x : p \land q \\
&\quad \vdash p \land q \vdash p \\
&\quad \vdash (p \land q) \rightarrow p
\end{align*}
\]
A Derivation

\[\begin{array}{c}
x : p \land q \vdash x : p \land q \\
x : p \land q \vdash \text{fst}(x) : p \\
\hline
\vdash (p \land q) \rightarrow p
\end{array}\]
A Derivation

\[
\begin{align*}
\frac{x : p \land q \vdash x : p \land q}{x : p \land q \vdash \text{fst}(x) : p} \\
\vdash \text{function } x \leftrightarrow \text{fst}(x) : (p \land q) \rightarrow p
\end{align*}
\]
A Derivation

\[
\begin{align*}
\frac{x : p \land q}{x : p \land q} & \quad \frac{x : p \land q}{\text{fst}(x) : p} \\
\frac{x : p \land q}{\text{function } x \mapsto \text{fst}(x) : (p \land q) \rightarrow p}
\end{align*}
\]

We write this function:

\[
\lambda x . \text{fst}(x)
\]
Lambda and Functions

- What's the derivative of $3xy^2$?

- With respect to $y$, its derivative is $6xy = \lambda y$.
- So if $x = 2$, that's the function $12y = \lambda y$.

- With respect to $x$, its derivative is $3y^2 = \lambda x$.
- So if $y = 3$, that's the function whose value is always $27 = \lambda x$.
Lambda and Functions

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But:
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But:

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- So if \(x = 2\), that’s the function \(12y\)

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Lambda and Functions

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  - With respect to $y$, its derivative is $6xy$
  - So if $x = 2$, that’s the function $12y$

- But:
  - With respect to $x$, its derivative is $3y^2$
  - So if $y = 3$, that’s the function whose value is always $27$

$$D(\lambda y \cdot 3xy^2) = \lambda y \cdot 6xy$$
$$D(\lambda x \cdot 3xy^2) = \lambda y \cdot 12y$$
What’s the derivative of $3xy^2$?

- With respect to $y$, its derivative is $6xy$.
- So if $x = 2$, that’s the function $12y$.

But:

- With respect to $x$, its derivative is $3y^2$.
- So if $y = 3$, that’s the function whose value is always $27$.

\[ D(\lambda y \cdot 3xy^2) = \lambda y \cdot 6xy \]
\[ = \lambda y \cdot 12y \]
\[ D(\lambda x \cdot 3xy^2) = \lambda x \cdot 3y^2 \]
What’s the derivative of $3xy^2$?

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- So if $x = 2$, that’s the function $12y$

But:

- With respect to $x$, its derivative is $3y^2$
- So if $y = 3$, that’s the function whose value is always $27$
A Derivation

\[
\begin{array}{c}
\frac{x : p \land q \vdash x : p \land q}{x : p \land q \vdash \text{fst}(x) : p} \\
\frac{x : p \land q \vdash \text{fst}(x) : p}{\vdash \lambda x . \text{fst}(x) : (p \land q) \to p}
\end{array}
\]
Implication Rules

\[ \Gamma, \varphi \vdash \psi \]
\[ \Gamma, \varphi \vdash \psi \]
\[ \Gamma \vdash \varphi \rightarrow \psi \]

\[ \Gamma \vdash \varphi \rightarrow \psi \]
\[ \Gamma \vdash \varphi \]
\[ \Gamma \vdash \psi \]
Implication Rules

\[
\begin{align*}
\Gamma, x : \varphi & \vdash s : \psi \\
\Gamma & \vdash \lambda x \cdot s : \varphi \rightarrow \psi
\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash \varphi \rightarrow \psi \quad \Gamma & \vdash \varphi \\
\hline
\Gamma & \vdash \psi
\end{align*}
\]
Implication Rules

\[ \Gamma, x : \varphi \vdash s : \psi \]
\[ \Gamma \vdash \lambda x . s : \varphi \rightarrow \psi \]

\[ \Gamma \vdash s : \varphi \rightarrow \psi \quad \Gamma \vdash t : \varphi \]
\[ \Gamma \vdash (st) : \psi \]
A Proof of an Implication is a Function
from proofs of the hypothesis
to proofs of the conclusion

\[ \vdash \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi} \]
\[ \vdash \frac{\Gamma \vdash \varphi}{\Gamma \vdash \psi} \]
A Proof of an Implication is a Function
from proofs of the hypothesis
to proofs of the conclusion

\[
\vdash d_1 : \ldots \\
\vdash d_2 \\
\Gamma, x : \varphi \vdash s : \psi \\
\Gamma \vdash \lambda x . s : \varphi \rightarrow \psi \\
\Gamma \vdash \varphi \\
\Gamma \vdash \psi
\]
A Proof of an Implication is a Function
to proofs of the conclusion

\[ \vdash x : \varphi \mid s : \psi \]
\[ \vdash \lambda x . s : \varphi \to \psi \]
\[ \vdash t : \varphi \]
\[ \vdash \psi \]
A Proof of an Implication is a Function
from proofs of the hypothesis
to proofs of the conclusion

\[
\Gamma, x: \varphi \vdash s: \psi \quad \vdots \\
\Gamma \vdash \lambda x. s: \varphi \rightarrow \psi \\
\Gamma \vdash t: \varphi \\
\Gamma \vdash (\lambda x. s)t: \psi
\]
A Proof of an Implication is a Function
from proofs of the hypothesis
to proofs of the conclusion

\[
\vdash \frac{\Gamma, x : \varphi \vdash s : \psi}{\Gamma, x : \varphi \vdash \lambda x . s : \varphi \rightarrow \psi} \quad \vdash \frac{t : \varphi}{s[t/x] : \psi}
\]
Disjunction Rules

\[ \Gamma \vdash \varphi \]
\[ \Gamma \vdash \varphi \lor \psi \]

\[ \Gamma \vdash \psi \]
\[ \Gamma \vdash \varphi \lor \psi \]

\[ \Gamma \vdash \varphi \lor \psi \]
\[ \Gamma, \varphi \vdash \chi \]
\[ \Gamma, \psi \vdash \chi \]
\[ \Gamma \vdash \chi \]
Disjunction Rules

\[\Gamma \vdash s : \varphi\]
\[\Gamma \vdash \langle \text{lft, } s \rangle : \varphi \lor \psi\]
\[\Gamma, \varphi \vdash \chi\]
\[\Gamma, \psi \vdash \chi\]

\[\Gamma \vdash \varphi \lor \psi\]

\[\Gamma \vdash \psi\]

\[\Gamma \vdash \varphi \lor \psi\]

\[\Gamma \vdash \chi\]
Disjunction Rules

\[ \Gamma \vdash s : \varphi \]
\[ \Gamma \vdash \langle \text{lft, } s \rangle : \varphi \lor \psi \]
\[ \Gamma \vdash s : \psi \]
\[ \Gamma \vdash \langle \text{rgt, } s \rangle : \varphi \lor \psi \]

\[ \Gamma \vdash \varphi \lor \psi \quad \Gamma, \varphi \vdash \chi \quad \Gamma, \psi \vdash \chi \]
\[ \Gamma \vdash \chi \]
Disjunction Rules

\[
\begin{align*}
\Gamma & \vdash s : \varphi \\
\Gamma & \vdash \langle \text{lft}, s \rangle : \varphi \lor \psi \\
\Gamma & \vdash \langle \text{rgt}, s \rangle : \varphi \lor \psi
\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash s : \varphi \lor \psi \\
\Gamma, x : \varphi & \vdash t : \chi \\
\Gamma, y : \psi & \vdash r : \chi
\end{align*}
\]

\[
\Gamma \vdash \text{cases}(s, \lambda x . t, \lambda y . r) : \chi
\]
Another Derivation

\[
\begin{align*}
(p \land q) & \vdash p \land q \\
(p \land q) & \vdash p \\
(p \land q) & \vdash p \lor q \\
(p \land q) \to (p \lor q)
\end{align*}
\]
Another Derivation

\[\begin{align*}
\text{x: } p \land q & \vdash \text{x: } p \land q \\
\hline
p \land q & \vdash p \\
\hline
p \land q & \vdash p \lor q \\
\hline
(p \land q) & \rightarrow (p \lor q)
\end{align*}\]
Another Derivation

\[
\begin{align*}
  x: p \land q & \vdash x: p \land q \\
  x: p \land q & \vdash \text{fst}(x): p \\
  p \land q & \vdash p \lor q \\
  (p \land q) & \rightarrow (p \lor q)
\end{align*}
\]
Another Derivation

\[
\begin{array}{c}
x : p \land q \vdash x : p \land q \\
x : p \land q \vdash \text{fst}(x) : p \\
x : p \land q \vdash \langle \text{lft}, \text{fst}(x) \rangle : p \lor q \\
\hline
(p \land q) \rightarrow (p \lor q)
\end{array}
\]
Another Derivation

\[ \begin{align*}
  x : p \land q & \vdash x : p \land q \\
  x : p \land q & \vdash \text{fst}(x) : p \\
  x : p \land q & \vdash \langle \text{lft}, \text{fst}(x) \rangle : p \lor q \\
  \lambda x . \langle \text{lft}, \text{fst}(x) \rangle & : (p \land q) \rightarrow (p \lor q)
\end{align*} \]
Axiom and Bottom

\[ \Gamma, x : \varphi \vdash x : \varphi \]

\[ \Gamma \vdash \bot \]

\[ \Gamma \vdash \varphi \]
The Empty Promise Rule

\[ \Gamma \vdash x: \varphi \]
\[ \Gamma \vdash x: \bot \]

\[ \Gamma, x: \varphi \vdash x: \varphi \]
\[ \Gamma \vdash x: \bot \]

\[ \Gamma \vdash \varphi \]
Axiom and Bottom

\[ \Gamma \vdash x : \varphi \]

\[ \Gamma, x : \varphi \vdash x : \varphi \]

\[ \Gamma \vdash \text{emp}(x) : \varphi \]
Axiom and Bottom

\[ \Gamma, x : \phi \vdash x : \phi \quad \Gamma \vdash \text{emp}(x) : \phi \]

The Empty Promise Rule
Axiom and Bottom

![Axiom and Bottom Diagram]

The Wishful Thinking Rule

\[ \Gamma, x : \varphi \vdash x : \varphi \]

\[ \Gamma \vdash \text{emp}(x) : \varphi \]
Wishful Thinking

What about \( \text{fst}(x) \)?

\[
\begin{align*}
\frac{x : p \land q}{x : p \land q} & \vdash x : p \land q \\
\frac{x : p \land q}{x : p \land q} & \vdash \text{fst}(x) : p \\
\frac{x : p \land q}{x : p \land q} & \vdash \langle \text{lft}, \text{fst}(x) \rangle : p \lor q \\
\lambda x . \langle \text{lft}, \text{fst}(x) \rangle & : (p \land q) \to (p \lor q)
\end{align*}
\]
Wishful Thinking

What about $\text{fst}(x)$?

\[
\frac{x : p \land q \vdash x : p \land q}{x : p \land q \vdash \text{fst}(x) : p}
\]

\[
\frac{x : p \land q \vdash \langle \text{lft}, \text{fst}(x) \rangle : p \lor q}{\lambda x . \langle \text{lft}, \text{fst}(x) \rangle : (p \land q) \rightarrow (p \lor q)}
\]

This is like typechecking
Propositions as types

- A proposition is a type
- A proof ("construction") is an object of that type
- The rules must ensure:
  
  If a value $s$ can be bound to a variable $x$
  and $x$ occurs in an elimination context, e.g. $\text{fst}(x)$
  then $s$ permits that operation, e.g. $s = \langle t_1, t_2 \rangle$
Propositions as types

- A proposition is a type
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  If a value $s$ can be bound to a variable $x$
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  then $s$ permits that operation, e.g. $s = \langle t_1, t_2 \rangle$

What are

- the computational rules for manipulating these objects?
- the properties ensured by those rules?