# Büchi Automata and Linear Temporal Logic

Joshua D. Guttman

Worcester Polytechnic Institute

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# Büchi Automata

## Definition

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$\mathcal{A} = \langle Q, \Sigma, \delta, Q_s, F \rangle$ where:

- $Q$ is a finite set called the *states*
- $\Sigma$ is a finite set called the *alphabet*
- $\delta \subseteq Q \times \Sigma \times Q$, is called the *transition relation*
- $Q_s \subseteq Q$ is called the set of *initial states*
- $F \subseteq Q$ is called the set of *accepting states*

# Büchi Automata: Runs, acceptance, language

Let $W$ be an infinite sequence $\langle w_0, w_1, \ldots \rangle$ with $w_i \in \Sigma$

- A *run $R$ of $\mathcal{A}$ for $W$*: an infinite sequence $\langle r_0, r_1, \ldots \rangle$ with $r_i \in Q$ where
  1. $r_0 \in Q_s$
  2. $(r_i, w_i, r_{i+1}) \in \delta$

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$$\{i \colon r_i \in F\} \quad \text{is infinite}$$

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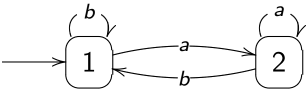
- The *language accepted by $\mathcal{A}$*, written $\mathrm{lang}(\mathcal{A})$, is

$$\{W : \mathcal{A} \text{ accepts } W\}$$

# Some Büchi Automata, 1

$\Sigma = \{a, b\}$

$$F = \{2\}$$



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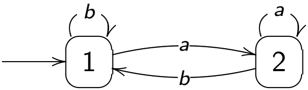
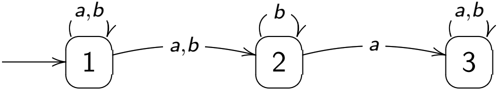



Accepts $W$ iff $W$ has infinitely many $a$s

# Some Büchi Automata, 2

$\Sigma = \{a, b\}$

$$F = \{2\}$$



# Some Büchi Automata, 2

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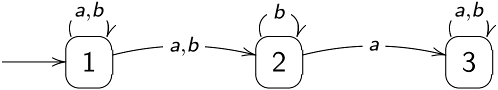


Accepts $W$ iff $W$ has finitely many $a$s

Can transition to state 2 after last $a$ received

# Testing Non-Emptiness

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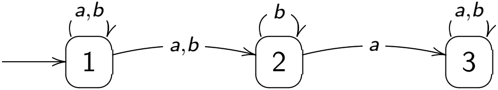

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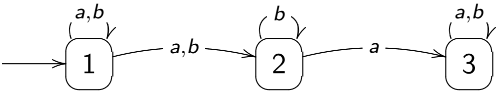


There exists a $W \in \text{lang}(\mathcal{A})$ if there is a $q$ with

1. a path from a start state to $q$
2. a cycle $q \rightarrow^{+} q$

# Closure Conditions on Languages

Suppose given Büchi automata $\mathcal{A}_1$, $\mathcal{A}_2$

there exist Büchi automata with languages:

$$\mathsf{lang}(\mathcal{A}_1) \cup \mathsf{lang}(\mathcal{A}_2)$$

$$\mathsf{lang}(\mathcal{A}_1) \cap \mathsf{lang}(\mathcal{A}_2)$$

$$\Sigma^\omega \setminus \mathsf{lang}(\mathcal{A}_1)$$

# Büchi Automata Representing LTL formula $\varphi$

$\varphi$ contains atomic formulas $\mathcal{L}$

$\Sigma$: Input letters are prop. logic models $\mathbb{M} \subseteq \mathcal{L}$

$Q$: Each subformula of $\varphi$ is a state, plus $\bot, \top, \ldots$

$\delta(\psi, \mathbb{M})$: depends on form of $\psi$:

$$\begin{array}{ll} \top, & \text{if } \psi \text{ in prop. logic and } \mathbb{M} \models \psi \\ \bot, & \text{if } \psi \text{ in prop. logic and } \mathbb{M} \not\models \psi \end{array}$$

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if $\psi$ is $\alpha U \beta$,

$$\delta(\beta, \mathbb{M}) \vee (\delta(\alpha, \mathbb{M}) \wedge (\alpha U \beta))$$

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$F$: $\{\psi \in Q \colon \psi = \neg(\alpha U \beta)\}$

# Model Checking, 1: What?

Suppose we are given:

- A Kripke structure $\mathcal{K}$
  - Represents a system
- An LTL formula $\varphi$
  - Represents a specification
    "correctness condition"

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satisfy $\varphi$?

For every $\pi$ over graph $\mathcal{K}$,
does $\pi \models \varphi$ hold?

# Model Checking, 2: How?

Let $\alpha$ be an LTL formula over atoms $\mathcal{L}$
Let $\mathcal{K} = (S, I, T, L)$ be a finite Kripke structure with $L(s) \subseteq \mathcal{L}$

- $\neg\alpha$ determines an automaton $\mathcal{A}$

# Model Checking, 2: How?

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- $\neg\alpha$ determines an automaton $\mathcal{A}$
- $\mathcal{K}$ determines a Büchi automaton

$$\mathcal{B} = S', \Sigma, \delta, I, S$$

where $S' = S \cup \{\text{fail}\}$ $\Sigma = 2^{\mathcal{L}}$, and

$$\begin{aligned} \delta(s, \mathbb{M}) &= \{s' \in T(s) \colon L(s) = \mathbb{M}\} \\ &\cup \{\text{fail} \colon s = \text{fail or } L(s) \neq \mathbb{M}\} \end{aligned}$$

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- $\emptyset \stackrel{?}{=} \mathrm{lang}(\mathcal{A}) \cap \mathrm{lang}(\mathcal{B})$