Consequence Relations and Natural Deduction

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1 Consequence Relations

A useful notion that cuts across both semantic (model-oriented) and syntactic (derivation-oriented) issues is the notion of a consequence relation. We will use capital Greek letters like Γ, Δ (Gamma and Delta) to refer to finite sets of formulas, and lower case Greek letters like φ, ψ (phi and psi) to refer to individual formulas. We will save ink by writing Γ, Δ for the set Γ ∪ Δ, and Γ, φ for the set Γ ∪ {φ}, etc.

By φ[t_1/x_1, \ldots, t_n/x_n], we mean the result of plugging in the terms t_1, \ldots, t_n in place of the variables x_1, \ldots, x_n. We assume that all the x_i are different variables, and that all of the plugging in happens at once. So, if there are x_2s inside the term t_1, they are not substituted with t_2s. Γ[t_1/x_1, \ldots, t_n/x_n] means the result of doing the substitutions to all the formulas in Γ.

Definition 1 Suppose that ≤ is a relation between finite sets of formulas and individual formulas, as in Γ ≤ φ. Then ≤ is a consequence relation iff it satisfies these properties:

Reflexivity: Γ, φ ≤ φ;

Transitivity: Γ ≤ φ and Γ, φ ≤ ψ imply Γ ≤ ψ;
Weakening: $\Gamma \preceq \phi$ implies $\Gamma, \Delta \preceq \phi$; and

Substitution: $\Gamma \preceq \phi$ implies $\Gamma[t_1/x_1, \ldots t_n/x_n] \preceq \phi[t_1/x_1, \ldots t_n/x_n]$.

For now, we will focus on formulas with no variables, so Substitution will be irrelevant. We will ignore it until later. The Reflexivity and Transitivity rules ensure that a consequence relation is a partial order, when restricted to sets containing just one assumption. The Weakening rule “lifts” this partial order to sets with more members.

We refer to an instance of a relation $\Gamma \preceq \phi$ or any $\Gamma R \phi$ as a judgment.

Both semantic notions such as entailment and syntactic notions such as derivability give us examples of consequence relations. Suppose we have a notion of model such as $M \models \phi$ as defined in the Dougherty lecture notes, Def. 2.2.2. Then we have a corresponding notion of (semantic) entailment defined:

**Definition 2** $\Gamma$ entails $\phi$, written $\Gamma \Vdash \phi$, holds iff, for all models $M$:

If for each $\psi \in \Gamma$, $M \models \psi$,

then $M \models \phi$.

That is, $\Gamma \Vdash \phi$ means that every model that makes all of the formulas in $\Gamma$ true makes $\phi$ true too.

**Lemma 3** Entailment is a consequence relation, i.e. $\Vdash$ satisfies reflexivity, transitivity, and weakening in Def. 7:

1. $\Gamma, \phi \Vdash \phi$;
2. $\Gamma \Vdash \phi$ and $\Gamma, \phi \Vdash \psi$ imply $\Gamma \Vdash \psi$; and
3. $\Gamma \Vdash \phi$ implies $\Gamma, \Delta \Vdash \phi$.

We turn next to showing that a particular set of rules for constructing proofs is also a consequence relation.

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1Available at URL [http://web.cs.wpi.edu/~guttman/cs521_website/Dougherty_lecture_notes.pdf](http://web.cs.wpi.edu/~guttman/cs521_website/Dougherty_lecture_notes.pdf)
2 A Derivation System for “Natural Deduction”

We consider the rules suggested by Gerhart Gentzen as a “natural” form of deduction. Gentzen considered these rules natural because they seemed to match directly the meaning of each logical operator.

Each logical operator has one or a couple of rules that allow you to prove formulas containing it as the outermost operator. These are called introduction rules. Each operator also has one or a couple of rules that allow you to prove other formulas by extracting the logical content in a formula containing it as outermost operator. They are called elimination rules. The introduction rules push formulas up in the partial ordering, while the elimination rules hold them down. Between them, the introduction and elimination rules fix the meaning of the logical operators purely in terms of their deductive power.

All of this extends to much richer logics, as we will see.

The rules are spread out through Figs. 1-4.

Definition 4 A natural deduction derivation is a tree, conventionally written with the conclusion, the root, at the bottom, such that each judgment is the conclusion of a rule.

A derivation is a natural deduction derivation in intuitionist propositional logic if each rule is one of those shown in Figs. 1–4.

An example derivation is shown in Fig. 5. It proves ⊢ (p ∧ q) → (p ∨ q).

There are two questions we’d immediately like answers to. First, do the derivable judgments form a consequence relation? That is, if Γ ⪯ φ means that there is a derivation of Γ ⊢ φ using our rules, then is ⪯ a consequence relation?

Second, how do derivable judgments relate to entailment? If Γ ⊢ φ is derivable, then is Γ ⊩ φ true? If Γ ⊩ φ then is Γ ⊢ φ derivable?

We can answer the first question affirmatively.

Lemma 5 The set of derivable judgments Γ ⊢ φ form a consequence relation.

Proof: 1. **Reflexivity** holds because Γ, φ ⊢ φ is always a derivation.

2. **Transitivity** holds by Fig. 6.

3. Weakening holds by *induction on derivations*:
**Base Case** Suppose that there is a derivation of $\Gamma \vdash \phi$ consisting only of an application of the Axiom rule. That is, $\phi \in \Gamma$. Thus, $\phi \in \Gamma, \Delta$, so $\Gamma, \Delta \vdash \phi$ is an application of the Axiom rule.

**Induction Step** Suppose that we are given a derivation $d$ where the last step is an application of one of the rules from Figs. 1–4, and the previous steps generate one or more subderivations $d_i$, each with conclusion $\Gamma_i \vdash \psi_i$.

*Induction hypothesis.* Assume that for each of the subderivations $d_i$, there is a weakened subderivation $W(d_i)$ such that $W(d_i)$ has conclusion $\Gamma_i, \Delta \vdash \psi_i$.

Construct the desired derivation of $\Gamma, \Delta \vdash \phi$ by combining the weakened subderivations $W(d_i)$ using the same rule of inference.

$\square$

One part of the second question is easy to answer.

**Lemma 6** $\vdash \subseteq \models$.

*That is, if $\Gamma \vdash \phi$ is derivable, then $\Gamma \models \phi$.*

**Proof:** By induction on derivations. $\square$

On the other hand, $\vdash \subsetneq \models$. There are entailment relations that cannot be derived using these rules.

**Challenge.** Find a $\Gamma, \phi$ such that $\Gamma \models \phi$ but $\Gamma \vdash \phi$ is not derivable using our rules. How would one prove it not derivable?

**Question.** If these rules do not characterize the semantic entailment relation generated from the classical $\models$, what do they characterize?