Type Preservation and Normalization: Two Consequences

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Today’s Goal

Theorem
Every normal derivation has the subformula property

Corollary
The derivation rules are consistent: there is no derivation \( \Gamma \vdash s : \bot \)

Subformula Property

A derivation of proving \( \Gamma \vdash s : \varphi \) has the subformula property iff,
for every \( \Delta \vdash t : \psi \) appearing in \( d \),
either \( \psi \) is a subformula of \( \varphi \),
or \( \psi \) is a subformula of \( \chi \),
where some \( \chi : \chi \in \Gamma \)

Subformula is transitive, and:
\( \bot \) is a subformula of every \( \varphi \)
\( \varphi \) is a subformula of \( \varphi \)
\( \varphi, \psi \) are subformulas of:
\( \varphi \land \psi, \varphi \lor \psi, \varphi \rightarrow \psi \)

Proving Consistency
From the Subformula Property

Corollary
The derivation rules are consistent: there is no derivation \( \Gamma \vdash s : \bot \).

Proof.
\( \Gamma \vdash s : \bot \) is not an instance of an axiom.
It may be derived by the \( \bot \) rule, if \( s = \text{emp}(t) \), but then there is no progress, as the premise is again of the form \( \Gamma \vdash t : \bot \).
Every other rule requires using a formula that is not a subformula of \( \bot \).

Two Computational Theorems
Type Preservation and Normalization

Theorem (Type Preservation)
If \( s \xrightarrow{*} t \) and \( \Gamma \vdash s : \varphi \), then also \( \Gamma \vdash t : \varphi \).

Theorem (Normal Form)
If \( \Gamma \vdash s : \varphi \), then there is a normal form \( t \) such that \( \Gamma \vdash s \xrightarrow{*} t \).
Local reduction rules

\[ \begin{align*}
\text{fst}(s', t') & \rightarrow r s' \\
\text{scd}(s', t') & \rightarrow s' \\
\text{cases}(\text{left}, s, t) & \rightarrow t s \\
\text{cases}(\text{right}, s, t) & \rightarrow r s \\
(\lambda v . s) t & \rightarrow s[t/v] \\
\end{align*} \] (β)

Reducing Intro/Elim Pair: \( \rightarrow \)

\[ \begin{align*}
\Gamma & \vdash s : \varphi \\
\Gamma & \vdash t : \psi \\
\Gamma & \vdash s[t/x] : \psi \\
\end{align*} \]

Reducing Intro/Elim Pair: \( \land \)

\[ \begin{align*}
\Gamma & \vdash s : \varphi \\
\Gamma & \vdash t : \psi \\
\Gamma & \vdash s[t/x] : \psi \\
\end{align*} \]

Compile-time reduction rules

\[ \begin{align*}
\text{fst}(\text{cases}(s, t, r)) & \rightarrow r \text{cases}(\text{fst}(s), t, \text{fst}(r)) \\
\text{scd}(\text{cases}(s, t, r)) & \rightarrow r \text{cases}(s, \text{fst}(t), \text{scd}(r)) \\
\text{cases}(\text{cases}(s, t, r), u, w) & \rightarrow r \text{cases}(s, \lambda r . \text{cases}(u, r), w) \\
(u \text{cases}(s, t, r)) & \rightarrow r \text{cases}(s, u \circ t, u \circ r) \\
\end{align*} \]

Reducing a cases/elim pair: \( \lor \)

\[ \begin{align*}
\Gamma & \vdash s : \varphi \lor \psi \\
\Gamma & \vdash t : \chi_1 \lor \chi_2 \\
\Gamma & \vdash y : \psi \lor r : \chi_1 \land \chi_2 \\
\Gamma & \vdash \text{cases}(u, \lambda r . \text{cases}(v, r), w) \\
\Gamma & \vdash \text{cases}(u, \lambda r . \text{cases}(v, r), w) \\
\end{align*} \]

Omitted: Analogous deriv. of \( \Gamma, y : \varphi \vdash g_2 : \rho \)

\[ \begin{align*}
\Gamma & \vdash s : \varphi \lor \psi \\
\Gamma & \vdash t : \chi_1 \lor \chi_2 \\
\Gamma & \vdash y : \psi \lor r : \chi_1 \land \chi_2 \\
\Gamma & \vdash \text{cases}(u, \lambda r . \text{cases}(v, r), w) \\
\Gamma & \vdash \text{cases}(u, \lambda r . \text{cases}(v, r), w) \\
\end{align*} \]
### The Reduction Relation

- \( s \rightarrow t \)
- \( s \rightarrow^* t \)
- \( C[s] \rightarrow C[t] \)
- \( s \rightarrow^* t \rightarrow u \)
- \( s \rightarrow^* u \)

### Contexts \( C[x] \)

Replace any \( s, t, u \) with an \( x \) to make a context \( C[x] \):

\[
C[x] ::= x | \langle C'[x], t \rangle | \langle s, C'[x] \rangle | \text{scd}(C'[x]) |
\]

### Two Computational Theorems

#### Type Preservation and Normalization

### Theorem (Type Preservation)

If \( s \rightarrow^* t \) and \( \Gamma \vdash s : \phi \), then

- also \( \Gamma \vdash t : \phi \).

### Theorem (Normal Form)

If \( \Gamma \vdash s : \phi \), there is a normal form \( t \) such that \( s \rightarrow^* t \).

### A Normal Proof

\[
\begin{align*}
p, (p \rightarrow \bot) \land q & \vdash (p \rightarrow \bot) \land q \quad p, (p \rightarrow \bot) \land q & \vdash p \quad p, (p \rightarrow \bot) \land q & \vdash (p \rightarrow \bot) \\
p, (p \rightarrow \bot) \land q & \vdash q \\
(p \rightarrow \bot) \land q & \vdash (p \rightarrow \bot) \\
& \vdash ((p \rightarrow \bot) \land q) \rightarrow (p \rightarrow q)
\end{align*}
\]

\[\lambda x. \lambda y. \text{emp}((\text{fst}(x), y))\]

### Another Normal Proof

\[
\begin{align*}
p, (p \lor q) \rightarrow r & \vdash (p \lor q) \rightarrow r \quad p, (p \lor q) & \vdash r \quad p \lor q & \vdash (p \lor q) \\
p \lor q & \vdash (p \lor q) \rightarrow (p \rightarrow r) \quad \lambda y. \lambda x. (y \langle \text{fst}(x) \rangle)
\end{align*}
\]
Normal Derivations, 1

Regarding $s$ as a tree with the conclusion at the root

If $d$ is a normal derivation, then

- working upward from any point through major premises,
- every application of an introduction rule
- is reached before
- any application of an elimination rule

All premises in an introduction rule are major

The major premise of an elimination rule is the premise containing the connective to be eliminated

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Major and Minor Premises: Conjunction

\[
\Gamma \vdash s : \varphi \\
\Gamma \vdash t : \psi \\
\Gamma \vdash (s,t) : \varphi \land \psi
\]

\[
\Gamma \vdash s : \varphi \land \psi \\
\Gamma \vdash \text{fst}(s) : \varphi \\
\Gamma \vdash \text{scd}(s) : \psi
\]

---

Major and Minor Premises: Implication

\[
\Gamma, x : \varphi \vdash s : \psi \\
\Gamma \vdash \lambda x . s : \varphi \rightarrow \psi
\]

\[
\Gamma \vdash s : \varphi \rightarrow \psi \\
\Gamma \vdash t : \psi \\
\Gamma \vdash (s,t) : \psi
\]

\[
\Gamma \vdash s : \psi \\
\Gamma \vdash (\text{left},s) : \varphi \lor \psi \\
\Gamma \vdash (\text{right},s) : \varphi \lor \psi
\]

\[
\Gamma \vdash s : \varphi \lor \psi \\
\Gamma, x : \varphi \vdash t : \chi \\
\Gamma, y : \psi \vdash r : \chi \\
\Gamma \vdash \text{cases}(s, \lambda x . t, \lambda y . r) : \chi
\]

---

Major and Minor Premises: Disjunction

\[
\Gamma \vdash s : \varphi \\
\Gamma \vdash (\text{left},s) : \varphi \lor \psi \\
\Gamma \vdash (\text{right},s) : \varphi \lor \psi
\]

\[
\Gamma \vdash s : \varphi \lor \psi \\
\Gamma, x : \varphi \vdash t : \chi \\
\Gamma, y : \psi \vdash r : \chi \\
\Gamma \vdash \text{cases}(s, \lambda x . t, \lambda y . r) : \chi
\]

---

Major and Minor Premises: Axiom and Falsehood

\[
\Gamma, x : \varphi \vdash x : \varphi \\
\Gamma \vdash x : \bot \\
\Gamma \vdash \text{emp}(x) : \varphi
\]

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Normal Derivations, 1

Regarding $s$ as a tree with the conclusion at the root

If $d$ is a normal derivation, then

- working upward from any point through major premises,
- every application of an introduction rule
- is reached before
- any application of an elimination rule

All premises in an introduction rule are major

The major premise of an elimination rule is the premise containing the connective to be eliminated
Normal Derivations, 2

If \( d \) is a normal derivation, and \( p \) is any upwards path in \( d \)
if \( p \) traverses only elimination rules
and \( p \) traverses a disjunction elimination inference
then it is below any other elimination rule.
By the compile-time rules

Normal Derivations, 3

If \( d \) is a normal derivation, and \( p \) is any upwards path in \( d \)
if \( p \) traverses only elimination rules
then \( p \) traverses at most one major premise
of a disjunction elimination inference.
By the case/elim rule for \( \lor \)
and the previous claim

Normal Derivations, 4

If \( d \) is a normal derivation, and \( p \) is any upwards path in \( d \)
if \( p \) traverses only introduction rules
then each successive right hand side is a subformula of the one
below it.
By the form of the introduction rules

Reducing a cases/elim pair: \( \land \)

\[
\Gamma \vdash s : \phi \lor \psi \\
\Gamma, x : \phi \vdash t : \chi_1 \land \chi_2 \\
\Gamma, y : \psi \vdash r : \chi_1 \land \chi_2 \\
\Gamma \vdash \text{cases}(s, \lambda x . t, \lambda y . r) : \chi_1 \land \chi_2 \\
\Gamma \vdash \text{fst}(\text{cases}(s, \lambda x : t, \lambda y : r)) : \chi_1 \\
\Gamma, x : \phi \vdash t : \chi_1 \land \chi_2 \\
\Gamma, y : \psi \vdash r : \chi_1 \land \chi_2 \\
\Gamma \vdash \text{cases}(s, \lambda x : \text{fst}(t), \lambda y : \text{fst}(r)) : \chi_1 \\
\]

Omitted: Analogous deriv. of \( \Gamma \)

Reducing a cases/elim pair: \( \lor \)

\[
\Gamma \vdash s : \phi \lor \psi \\
\Gamma, x : \phi \vdash t : \chi_1 \lor \chi_2 \\
\Gamma, y : \psi \vdash r : \chi_1 \lor \chi_2 \\
\Gamma \vdash \text{cases}(s, \lambda x : t, \lambda y : r) : \chi_1 \lor \chi_2 \\
\Gamma \vdash t : r : \chi_1 \lor \chi_2 \\
\Gamma, x : \phi \vdash t : \chi_1 \lor \chi_2 \\
\Gamma, y : \psi \vdash r : \chi_1 \lor \chi_2 \\
\Gamma \vdash \text{cases}(s, \lambda x : \text{fst}(t), \lambda y : \text{fst}(r)) : \chi_1 \lor \chi_2 \\
\]

Today's Goal

Theorem

Every normal derivation has the subformula property
Why it’s true

**Theorem**
Every normal derivation $d$ of $\Gamma \vdash s : \varphi$ has the subformula property

**Proof.**
1. Conclusion of an introduction rule: subformula of $\varphi$
2. Major premise of an elimination rule: subformula of some $\psi$ in $\Gamma$
3. Minor premise of $\to$-elimination: subformula of the major premise
4. Minor premise of $\lor$-elimination: subformula of $\varphi$

Today’s Goal

**Theorem**
Every normal derivation has the subformula property

**Corollary**
The derivation rules are consistent:
there is no derivation $\vdash s : \bot$