

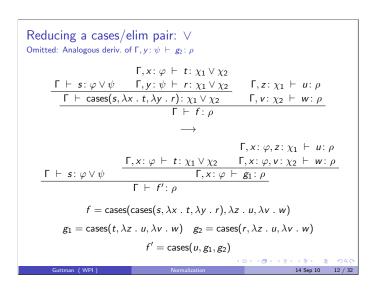
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Reducing a cases/elim pair: \land

\frac{\Gamma \vdash s : \varphi \lor \psi \qquad \Gamma, x : \varphi \vdash t : \chi_1 \land \chi_2 \qquad \Gamma, y : \psi \vdash r : \chi_1 \land \chi_2}{\Gamma \vdash \text{cases}(s, \lambda x \cdot t, \lambda y \cdot r) : \chi_1 \land \chi_2}{\Gamma \vdash \text{fst}(\text{cases}(s, \lambda x \cdot t, \lambda y \cdot r)) : \chi_1}

\frac{\Gamma, x : \varphi \vdash t : \chi_1 \land \chi_2}{\Gamma, x : \varphi \vdash \text{fst}(t) : \chi_1} \qquad \frac{\Gamma, y : \psi \vdash r : \chi_1 \land \chi_2}{\Gamma, y : \varphi \vdash \text{fst}(r) : \chi_1}

\frac{\Gamma \vdash s : \varphi \lor \psi \qquad \Gamma, x : \varphi \vdash \text{fst}(t) : \chi_1}{\Gamma, x : \varphi \vdash \text{fst}(t) : \chi_1} \qquad \frac{\Gamma, y : \psi \vdash r : \chi_1 \land \chi_2}{\Gamma, y : \varphi \vdash \text{fst}(r) : \chi_1}

\Gamma \vdash \text{cases}(s, \lambda x \cdot t, \lambda y \cdot r) : \chi_1 \land \chi_2 \land \chi_2 \land \chi_2 \land \chi_1 \land \chi_2 \land \chi_1 \land \chi_2 \land \chi_2 \land \chi_1 \land \chi_2 \land \chi_1 \land \chi_2 \land \chi_2 \land \chi_2 \land \chi_2 \land \chi_1 \land \chi_2 \land \chi_2 \land \chi_1 \land \chi_2 \land \chi_
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The Reduction Relation

$$\frac{s \longrightarrow_r t}{s \longrightarrow t} \qquad \frac{s \longrightarrow t}{\mathcal{C}[s] \longrightarrow \mathcal{C}[t]}$$

$$s \longrightarrow^* s \qquad \frac{s \longrightarrow^* t}{s \longrightarrow^* u}$$

Guttman (WPI)

lormalization

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Contexts C[x]

Replace any s, t, u with an x to make a context C[x]:

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Two Computational Theorems

Type Preservation and Normalization

Theorem (Type Preservation)

If $s \longrightarrow^* t$ and $\Gamma \vdash s : \varphi$, then also $\Gamma \vdash t : \varphi$.

Theorem (Normal Form)

If $\Gamma \vdash s \colon \varphi$, then there is a normal form t such that $s \longrightarrow^* t$.

Guttman (WPI

Normalization

on

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A Corollary: Normal Derivations

Corollary

- $\textbf{ 1} \ \textit{ If } \varphi \ \textit{ is derivable from } \Gamma \textit{, then} \\ \textit{ there is a normal derivation } t \textit{ such that } \Gamma \ \vdash \ t \colon \varphi$
- **1** If additionally $\Gamma = \emptyset$, then t is closed (i.e. no free variables)

Proof

- 1. If φ is derivable from Γ , then for some s, $\Gamma \vdash s \colon \varphi$. By normal form, $s \longrightarrow t$ for some normal t.
- By type preservation, $\Gamma \vdash t \colon \varphi$.
- 2. Via the Context Lemma, which says:

If $\Gamma \vdash s : \varphi$, then $fv(s) \subseteq dom(\Gamma)$.

Guttman (WPI)

Normalization

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A Normal Proof

$$\frac{p, (p \to \bot) \land q \vdash (p \to \bot) \land q}{p, (p \to \bot) \land q \vdash p \to \bot} \qquad p, (p \to \bot) \land q \vdash p}$$

$$\frac{p, (p \to \bot) \land q \vdash \bot}{p, (p \to \bot) \land q \vdash q}$$

$$\frac{p, (p \to \bot) \land q \vdash q}{(p \to \bot) \land q \vdash p \to q}$$

$$\vdash ((p \to \bot) \land q) \to (p \to q)$$

$$\lambda x \cdot \lambda y \cdot \text{emp}(\text{fst}(x) y)$$

Guttman (WP)

Normalization

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Another Normal Proof

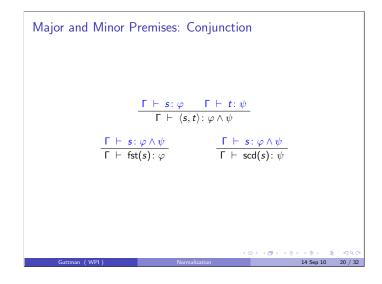
$$\frac{p,(p\vee q)\rightarrow r\ \vdash\ (p\vee q)\rightarrow r\ \stackrel{}{-}\frac{p,(p\vee q)\rightarrow r\ \vdash\ p}{p,(p\vee q)\rightarrow r\ \vdash\ p\vee q}}{\underbrace{\frac{p,(p\vee q)\rightarrow r\ \vdash\ r}{(p\vee q)\rightarrow r\ \vdash\ p\rightarrow r}}_{\vdash\ ((p\vee q)\rightarrow r)\rightarrow (p\rightarrow r)}$$

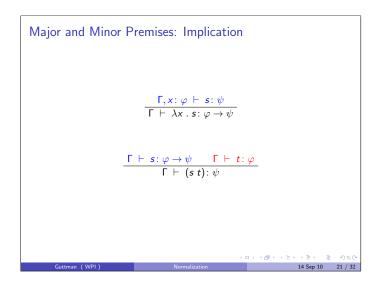
Guttman (WPI

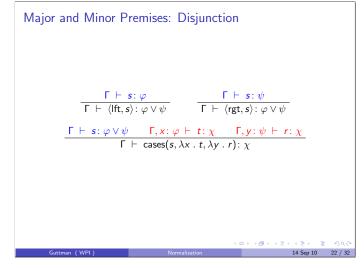
malization

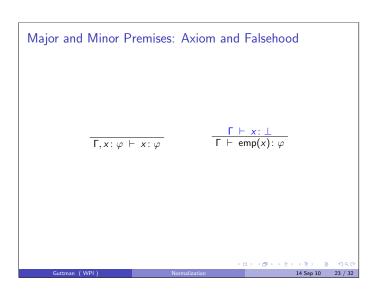
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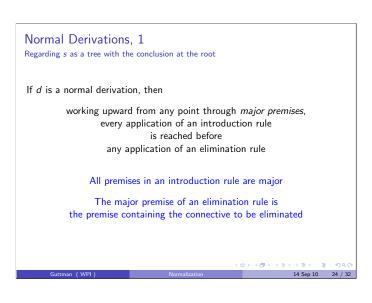
Normal Derivations, 1 Regarding s as a tree with the conclusion at the root If d is a normal derivation, then working upward from any point through major premises, every application of an introduction rule is reached before any application of an elimination rule All premises in an introduction rule are major The major premise of an elimination rule is the premise containing the connective to be eliminated











Normal Derivations, 2

If d is a normal derivation, and p is any upwards path in dif p traverses only elimination rules and p traverses a disjunction elimination inference then it is below any other elimination rule

By the compile-time rules

Reducing a cases/elim pair: ∨ Omitted: Analogous deriv. of $\Gamma, y \colon \psi \vdash g_2 \colon \rho$

Reducing a cases/elim pair: \land

 $\frac{\Gamma \vdash s \colon \varphi \lor \psi \qquad \Gamma, x \colon \varphi \vdash t \colon \chi_1 \land \chi_2 \qquad \Gamma, y \colon \psi \vdash r \colon \chi_1 \land \chi_2}{\Gamma \vdash \mathsf{cases}(s, \lambda x \cdot t, \lambda y \cdot r) \colon \chi_1 \land \chi_2}$

 $\frac{\Gamma \vdash s \colon \varphi \lor \psi \qquad \frac{\Gamma, x \colon \varphi \vdash t \colon \chi_1 \land \chi_2}{\Gamma, x \colon \varphi \vdash \mathsf{fst}(t) \colon \chi_1} \qquad \frac{\Gamma, y \colon \psi \vdash r \colon \chi_1 \land \chi_2}{\Gamma, y \colon \varphi \vdash \mathsf{fst}(r) \colon \chi_1}}{\Gamma \vdash \mathsf{cases}(s, \lambda x \cdot \mathsf{fst}(t), \lambda y \cdot \mathsf{fst}(r)) \colon \chi_1}$

 $\Gamma \vdash \mathsf{fst}(\mathsf{cases}(s, \lambda x \cdot t, \lambda y \cdot r)) \colon \chi_1$

 $\Gamma, x \colon \varphi, z \colon \chi_1 \vdash u \colon \rho$ $\frac{\Gamma, x \colon \varphi \vdash t \colon \chi_1 \lor \chi_2 \qquad \begin{array}{c} \Gamma, x \colon \varphi, z \colon \chi_1 \vdash u \colon \rho \\ \Gamma, x \colon \varphi \vdash t \colon \chi_1 \lor \chi_2 \qquad \Gamma, x \colon \varphi, v \colon \chi_2 \vdash w \colon \rho \end{array}}{\Gamma, x \colon \varphi \vdash g_1 \colon \rho}$

 $f = cases(cases(s, \lambda x . t, \lambda y . r), \lambda z . u, \lambda v . w)$

 $g_1 = cases(t, \lambda z . u, \lambda v . w)$ $g_2 = cases(r, \lambda z . u, \lambda v . w)$

 $f' = \operatorname{cases}(u, g_1, g_2)$

Normal Derivations, 3

If d is a normal derivation, and p is any upwards path in d

if p traverses only elimination rules

then p traverses at most one major premise of a disjunction elimination inference

By the case/elim rule for \lor and the previous claim

Normal Derivations, 4

If d is a normal derivation, and p is any upwards path in d

if p traverses only introduction rules

then each successive right hand side is a subformula of the one below it

By the form of the introduction rules

Today's Goal

Theorem

Every normal derivation has the subformula property

