Seven Rules for Big-O and Θ*

Here are seven rules that you can use to solve problems involving big-O and Θ. They will solve the big majority of the big-O and Θ comparisons you'll need in this course (and for a long way beyond). Two assumptions are noted in the *Fine Print* on the back.

$$\Theta(c \cdot f(x)) = \Theta(f(x)) \tag{1}$$

$$\Theta(f(x) + g(x)) = \Theta(\max(f(x), g(x))) \tag{2}$$

$$\Theta(f(x) \cdot h(x)) \leq \Theta(g(x) \cdot h(x)) \quad \text{if and only if} \quad \Theta(f(x)) \leq \Theta(g(x)) \tag{3}$$

$$\Theta(x^c) \leq \Theta(x^d) \quad \text{if and only if} \quad c \leq d \tag{4}$$

$$\Theta(\log x) < \Theta(x^c) \quad \text{if and only if} \quad 0 < c \tag{5}$$

Assuming that $c > 0$,

$$\Theta(x^c) < \Theta(d^x) \quad \text{if and only if} \quad 1 < d \tag{6}$$

Assuming that $1 \leq c$ and $1 \leq d$,

$$\Theta(c^x) < \Theta(d^x) \quad \text{if and only if} \quad c < d \tag{7}$$

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Fine Print. $\Theta(f)$ means the set of all functions g that grow essentially as fast as f. Officially, $\Theta(f) =$

$$\{g\text{: there exist } N_0, c_1, c_2 \text{ such that, for all } x > N_0,\\ g(x) \leq c_1 \cdot f(x) \text{ and } f(x) \leq c_2 \cdot g(x)\}.$$

So $\Theta(f) = \Theta(g)$, $g \in \Theta(f)$, and $f \in \Theta(g)$ all mean the same thing.

Big-O makes an ordering on the Θ-classes. By $\Theta(f) \leq \Theta(g)$, we mean that $f \in O(g)$. In fact, when $f \in O(g)$, either $f \in \Theta(g)$, or else every function $g' \in \Theta(g)$ asymptotically dominates every function $f' \in \Theta(f)$. So this ordering works in a compatible way across whole Θ-classes.

$\Theta(f) < \Theta(g)$ means $f \in O(g)$ but $f \notin \Theta(g)$.

A function f is *non-decreasing* if $x \leq y$ implies $f(x) \leq f(y)$. It's *eventually non-decreasing* if $N_0 < x \leq y$ implies $f(x) \leq f(y)$, for some N_0. A function is *eventually positive* if, for some N_0, for all $x > N_0$, $f(x) > 0$.

In the rules above, assume all the functions f, g are:

- eventually non-decreasing, and
- eventually positive.