An Algebra for Symbolic Diffie-Hellman Protocol Analysis

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Ephemeral DH

the optimistic view

$$
\begin{tikzcd}
A & \bullet \arrow[r, Rightarrow] \arrow[d, "g^x"] & \bullet \arrow[r, Rightarrow] & K_A \doteq (g^y)^x \\
B & \bullet \arrow[r, Rightarrow] & \bullet \arrow[r, Rightarrow] \arrow[u, "g^y"'] & K_B \doteq (g^x)^y
\end{tikzcd}
$$

$$K_A = (g^y)^x \;=\; (g^x)^y = K_B$$

in a cyclic group of prime order q

Amazing outcome: Shared secret via public information

Ephemeral DH

the realistic view

$$
\begin{tikzcd}
A & \bullet \arrow[r, Rightarrow] \arrow[d, "g^x"] & \bullet \arrow[r, Rightarrow] & K_A \doteq (R_B)^x \\
 & {} & {} \arrow[u, "R_B"'] & \\
 & {} \arrow[d, "R_A"] & {} & \\
B & \bullet \arrow[r, Rightarrow] & \bullet \arrow[r, Rightarrow] \arrow[u, "g^y"'] & K_B \doteq (R_A)^y
\end{tikzcd}
$$

If $R_A = g^x$ and $R_B = g^y$

then shared secret established

Ephemeral DH

the realistic view

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 & {} & {} \arrow[u, "R_B"'] & \color{red}{K_A \ \bullet} \\
 & {} \arrow[d, "R_A"] & {} & \color{red}{K_B \ \bullet} \\
B & \bullet \arrow[r, Rightarrow] & \bullet \arrow[r, Rightarrow] \arrow[u, "g^y"'] & K_B \doteq (R_A)^y
\end{tikzcd}
$$

If $R_A = g^{z}$ and $R_B = g^{w}$
where the adversary chose z, w

then K_A, K_B available to adversary

Security goal: Key secrecy

This diagram cannot occur

$$\bullet \Longrightarrow \bullet \Longrightarrow \circ \Longrightarrow \circ \; K_A$$

$$\bullet \; K_A$$

Subject to assumptions, e.g.
x, y randomly chosen by compliant principal

Static DH

Certificate authority authenticates, signs cert:

$$c_P = [\![\, \text{cert } Y_P,\ P \,]\!]_{\text{CA}}$$

where $Y_A = g^a \quad Y_B = g^b$

$$\begin{tikzcd}
A & \bullet \arrow[r, Rightarrow] \arrow[d, "c_A"] & \bullet \arrow[r, Rightarrow] & K_A \doteq (Y_B)^a = g^{ba} \\
B & \bullet \arrow[r, Rightarrow] & \bullet \arrow[r, Rightarrow] \arrow[u, "c_B"'] & K_B \doteq (Y_A)^b = g^{ab}
\end{tikzcd}$$

Drawback: A, B get same K for every run!

Implicitly Authenticated DH

- Use both ephemeral and static (certified) values
- Ephemeral g^x, g^y ensure variation in key
- Static g^a, g^b ensure authenticity implicitly:

If any principal P has computed K
then either $P = A$ or $P = B$

Security goal: Implicit authentication

$$[A, B, \ldots] \qquad \bullet \Longrightarrow \bullet \Longrightarrow \circ \Longrightarrow \circ\ K$$

$$[B', A', \ldots] \qquad \bullet \Longrightarrow \bullet \Longrightarrow \circ \Longrightarrow \circ\ K$$

If *long term values* a, b *unknown to adversary*
then $A = A'$, $B = B'$

This paper

1. Gives equational theory of abelian groups with exponentiation:

   AGˆ *characterizes the equations* $s = t$
   that are uniformly valid as group varies

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   Occurrences of secret exponents do not change
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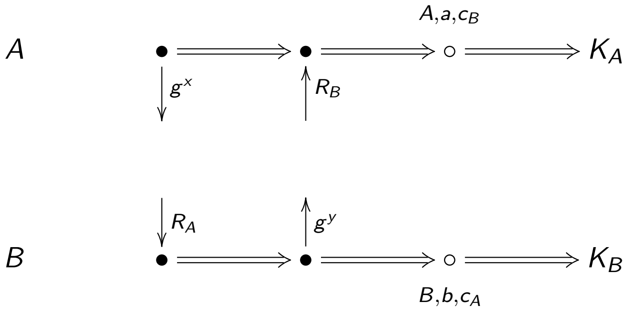
4. Shows security goals using indicator theorem

   Gives insights when they fail

IADH Protocols

$$c_A = [\![\, \text{cert } g^a,\ A \,]\!]_{\text{CA}} \qquad c_B = [\![\, \text{cert } g^b,\ B \,]\!]_{\text{CA}}$$



$$K_A = f(A, B, a, x, R_B, Y_B) \qquad K_B = f(A, B, b, y, R_A, Y_A)$$

Some IADH shared secret computations

Computation done by A

H($\cdot$) is a hash fn

$$K_{um} \quad = \quad \mathrm{H}({Y_B}^a, {R_B}^x) \quad \stackrel{?}{=} \qquad \mathrm{H}(g^{ab}, g^{xy})$$

Security goal: No impersonation

This diagram should be prevented

$$\bullet \Longrightarrow \bullet \Longrightarrow \circ \Longrightarrow \circ \; K_A$$

$$\bullet \; K_A$$

Your long term value b
and my ephemeral value x
unknown to adversary

$$K_A = \mathrm{H}(g^{ab}, g^{xy})$$

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\begin{aligned}
K_{um} &= \mathrm{H}({Y_B}^a, {R_B}^x) &\stackrel{?}{=}& \quad \mathrm{H}(g^{ab}, g^{xy}) \\
K_{cf} &= (R_B Y_B)^{x+a} &\stackrel{?}{=}& \quad g^{(y+b)(x+a)} = g^{xy} g^{ay} g^{xb} g^{ab}
\end{aligned}
$$

Our central contribution

Formal theory and semantics in which
occurrences of variables in exponents
are a security invariant

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K_{\_\_qv} &= (R_B {Y_B}^E)^{x+Da} &\stackrel{?}{=}& \quad g^{xy} g^{Day} g^{xEb} g^{DEab}
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UM = "Unified model" CF = Cremers-Feltz

MQV: $E = [R_B], \quad D = [g^x]$

HMQV: $E = \mathrm{H}(R_B, B), \quad D = \mathrm{H}(g^x, A)$

AGˆ

1. $(G, \cdot, inv, id)$ is an abelian group;
2. $(E, +, 0, -, *, 1)$ is a commutative ring with identity;
3. Exponentiation makes G a right E-module with identity:

$$
\begin{array}{lll}
(a^x)^y = a^{x * y} & a^1 = a & id^x = id \\
(a \cdot b)^x = a^x \cdot b^x & & a^{(x+y)} = a^x \cdot a^y
\end{array}
$$

4. Multiplicative inverse, closure at sort NZE, subsort of E:

$$
\begin{array}{lll}
u ** v = u * v & u * i(u) = 1 & i(-u) = -i(u) \\
i(u * v) = i(u) * i(v) & i(1) = 1 & i(i(w)) = w
\end{array}
$$

Rewriting relation $\rightarrow_{AG\hat{}}$

Theorem

The reduction $\rightarrow_{AG\hat{}}$ is terminating

Verified with the Aprove termination tool

Theorem

The reduction $\rightarrow_{AG\hat{}}$ is confluent mod AC

Verified with the Maude Church-Rosser checker

Free(AGˆ) as a message algebra

- Regular principals run protocol with values from Free(AGˆ)
  - Free choices are fresh variables a, b, x, y
  - Message sent/received are Free(AGˆ) terms over them
  - Augmented with encryption, signature, hashing, etc

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  - Free choices are fresh variables a, b, x, y
  - Message sent/received are Free(AGˆ) terms over them
  - Augmented with encryption, signature, hashing, etc
- Adversary model: can apply operations of Σ(AGˆ)
  - May multiply, add, take inverses, ...
  - No logarithms (:-)
  - May also encrypt and decrypt with key, pair, unpair
  - May choose variables unless assumed fresh
- Messages s, t are equal if AGˆ entails $s = t$

Indicators relative to secret vars

Indicator of a monomial m counts occurrences of these vars in m:

$$\mathrm{Ind}_{\langle a,b,x,y\rangle}(ab) = \langle 1,1,0,0\rangle \qquad \mathrm{Ind}_{\langle a,b,x,y\rangle}(xy) = \langle 0,0,1,1\rangle$$

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Indicators of $t_1 \cdot t_2$ union of indicators of t_1 and t_2

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$$\mathrm{Ind}_{\langle b,x\rangle}(g^{xy}g^{ay}g^{xb}g^{ab}) = \{\langle 0,1\rangle, \langle 1,1\rangle, \langle 1,0\rangle\}$$

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Indicators of pairs union of indicators etc

$$\mathsf{Ind}_{\langle b,x\rangle}(\mathsf{H}(g^{ab}, g^{xy})) = \{\langle 1,0\rangle, \langle 0,1\rangle\}$$

IADH Regular Behavior

For any basis $\vec{v}$

If t is any message sent by any compliant IADH participant,
then $\mathrm{Ind}_{\vec{v}}(t)$ is a basis vector

$$\langle \vec{0}, 1, \vec{0} \rangle$$

The indicator theorem

Theorem

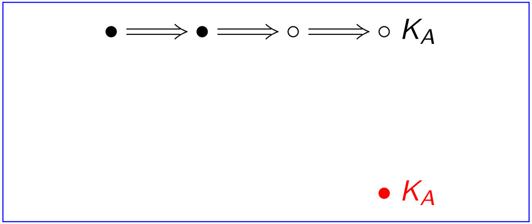
If the adversary can build t given messages S
then

$$\mathrm{Ind}_{\vec{v}}(t) \subseteq \bigcup_{s \in S} \mathrm{Ind}_{\vec{v}}(s) \cup \{\langle \vec{0} \rangle\}$$

when $\vec{v}$ is a list of secret NZE-variables

Security goal: Key secrecy

This diagram cannot occur in UM



$$K_A = \mathrm{H}(g^{ab}, {R_B}^{x})$$

Long term secrets a, b
uncompromised

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$$K_A = \mathsf{H}(g^{ab},\, {R_B}^{x})$$

Long term secrets a, b
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$$\langle 1, 1\rangle \in \mathsf{Ind}_{\langle a,b\rangle}(\mathsf{H}(g^{ab},\, {R_B}^{x}))$$

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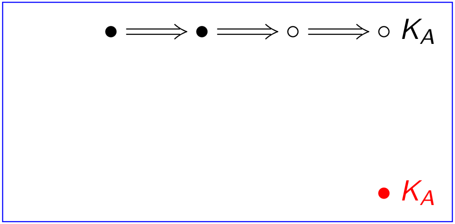

$$K_A = \mathrm{H}(g^{ab}, g^{xy})$$

Where your b and my x remain secret

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semantics

Mathematical context

DH structures

G a cyclic group of prime order q

g a generator of G

set E of exponents $\{0, 1, \ldots, (q-1)\}$ forms a field: $\mathbb{F}_q$

Useful group presentations for crypto include subgroups of

$\mathbb{Z}_p^*$ the integers mod p

elliptic curve over a finite field

Mathematical context

hard problems

1. Discrete Logarithm problem:

   given g^x*, compute* x

2. Computational Diffie-Hellman problem:

   given $g^x, g^y \in G$*, compute* g^{xy}

3. Decisional Diffie-Hellman problem:

   given $g^x, g^y \in G$*, distinguish* g^{xy} *from random* g^z

Considered intractable in suitable groups

Mathematical context

hard problems

1. Discrete Logarithm problem:

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3. Decisional Diffie-Hellman problem:

   given $g^x, g^y \in G$, distinguish g^{xy} from random g^z

Considered intractable in suitable groups
there is an infinite family of primes q s.t.
every PPT algorithm achieves advantage
only finitely often

Semantic requirement on AGˆ

- If $s = t$ valid for infinitely many q, adversary may use $s = t$
- Other tractable computations useful only in finitely many q

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- Equational completeness property:

$$\mathbb{F}_q \models s = t \text{ for infinitely many finite fields } \mathbb{F}_q$$

$$\text{implies}$$

$$\mathrm{AG\hat{}} \vdash s = t$$

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- If $s = t$ valid for infinitely many q, adversary may use $s = t$
- Other tractable computations useful only in finitely many q
- Equational completeness property:

$$\mathbb{F}_q \models s = t \text{ for infinitely many finite fields } \mathbb{F}_q$$
$$\text{implies}$$
$$\mathrm{AG\hat{}} \vdash s = t$$

- Actually: $\mathbb{F}_q \models s = t$ infinitely often iff $\mathrm{AG\hat{}} \vdash s = t$

Models of AGˆ, 1

For any field F, define $\mathcal{M}_F$ such that $\mathcal{M}_F \models$ AGˆ:

E, G both interpreted as dom(F)

NZE interpreted as dom(F) $\setminus \{0\}$

Operations of E interpreted as in F itself

$\cdot, inv, id$ interpreted as $+_F$, $-_F$, 0

a^e interpreted as $a * e$

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Some $\mathcal{M}_F$: When $F = \mathbb{F}_q$, we obtain $\mathcal{M}_q$
When $F = \mathbb{Q}$, we obtain $\mathcal{M}_{\mathbb{Q}}$

Models of AGˆ, 2

Let D be a non-principal ultrafilter over the prime numbers q

Write $\mathbb{F}_D$ for the ultraproduct

$$\prod_D \{\mathbb{F}_q \colon q \text{ prime}\}$$

Let $\mathcal{M}_D \models \text{AG}\hat{}$ be obtained from $\mathbb{F}_D$ as on last slide

Completeness of AGˆ for uniform equality

Theorem

For each pair of G-terms s and t, the following are equivalent

1. $\mathrm{AG\hat{}} \vdash s = t$
2. *For all q,* $\mathcal{M}_q \models s = t$
3. *For all non-principal D,* $\mathcal{M}_D \models s = t$
4. *For infinitely many q,* $\mathcal{M}_q \models s = t$
5. *For some non-principal D,* $\mathcal{M}_D \models s = t$
6. $\mathcal{M}_{\mathbb{Q}} \models s = t$
7. s, t *have the same normal form modulo AC*

This paper

1. Gives equational theory of abelian groups with exponentiation:
   - $\mathrm{AG}\hat{} \vdash s = t$ iff
     $\mathbb{F}_q \models s = t$ for infinitely many finite fields $\mathbb{F}_q$
   - Convergent associative-commutative rewriting system
   - Symbolic algebra Free(AGˆ) of normal forms
2. Formalizes implicitly authenticated Diffie-Hellman protocol behavior and adversary over Free(AGˆ)
3. Shows indicator theorem:

   Occurrences of secret exponents do not change through adversary actions

4. Shows security goals using indicator theorem

   Gives insights when they fail

A Handy Lemma about $\mathcal{M}_{\mathbb{Q}}$

Lemma

1. *$\mathcal{M}_{\mathbb{Q}}$ can be embedded as a submodel in any $\mathcal{M}_D$.*
2. *If s and t are distinct normal forms then $\mathcal{M}_{\mathbb{Q}} \not\models s = t$.*

Ultraproducts

D is an ultrafilter iff D is a maximal family of sets $\subseteq X$ such that:

- $\emptyset \notin D$
- $s_1, s_2 \in D$ implies $s_1 \cap s_2 \in D$
- $s_1 \in D$ and $s_1 \subseteq s_2$ implies $s_2 \in D$

D is principal iff $D = \{s \colon s_0 \subseteq s\}$ for some s_0

Ultraproduct $\prod_D \mathcal{M}_q$, for ultrafilter D:

let $\mathcal{M}_q$ be a family of structures indexed by $q \in X$
$\prod_D \mathcal{M}_q$ is a factored product such that

$$\prod_D \mathcal{M}_q \models \phi \quad \text{iff} \quad \{q \in X \colon \mathcal{M}_q \models \phi\} \in D$$

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Only consider non-principal ultrafilters