Disconnected colors in generalized Gallai colorings - research project

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Abstract

Gallai-colorings of complete graphs - edge colorings such that no triangle is colored with three distinct colors - occur in various contexts such as the theory of partially ordered sets (in Gallai’s original paper), information theory and the theory of perfect graphs. A basic property of Gallai-colorings with at least three colors is that at least one of the color classes must span a disconnected graph. We are interested here to find whether this or similar property remains true if we consider colorings that do not contain a multicolored copy of a fixed graph $F$.

1 Introduction

Edge colorings of complete graphs in which no triangle is colored with three distinct colors were called Gallai-partitions in [7], Gallai-colorings in [4], [5]. Here we call briefly these colorings as $G$-colorings and always assume that $G$-colorings are on the edges of a complete graph. More than just the term, the concept occurs in relation of deep structural properties of fundamental objects. An important result, Theorem 1, from Gallai’s original paper [3] – translated to English and endowed by comments in [8] – can be reformulated in terms of Gallai-colorings. Further occurrences are related to generalizations of the perfect graph theorem [1], or applications in information theory [6].

Our starting point is the following result of Gallai [3], see an explicit proof in [4].

**Theorem 1.** In every $G$-coloring with at least three colors, some of the color classes span a disconnected graph.

What is the role of forbidding a multicolored triangle? Can we extend some way Theorem 1 to colorings where a multicolored copy of some fixed graph $F$ is forbidden?
This question is the central topic. An edge coloring of a complete graph $K$ is connected if each color class has a spanning tree in $K$.

Let us say that a graph $F$ has the disconnection property, $DP$, if there exists a natural number $m = m(F)$ such that the following holds: in every edge coloring of a complete graph with at least $m$ colors either there is a multicolored $F$ or at least one color class is disconnected. Equivalently, $F$ has the disconnection property if in every connected coloring with $m(F)$ colors there is a multicolored copy of $F$. Notice that $m(F) \geq |E(F)|$ because large enough complete graph have connected colorings with $|V(F)| - 1$ colors. If a graph $F$ has $DP$ with $m(F) = |E(F)|$ we say that it has the Gallai property, $GP$. Sometimes $GP$ and $DP$ are just identified with the class of graphs having the property. Then we can say for example that by Theorem 1, $K_3 \in GP$.

**Observation 1.** If $F \in GP$ then $F \in DP$.

**Observation 2.** Assume that $F_1 \subset F_2$ and $F_2 \in DP$. Then $F_1 \in DP$.

**Proposition 1.** $P_3, P_4, P_5 \in GP$.

**Proof.** The result is trivial for $P_3$ and almost trivial for $P_4$ so we show only that $P_5$ is a Gallai-type graph. Let $G$ be a graph whose edges are colored with $m \geq 4$ colors and assume all color classes span connected graphs. We shall reach a contradiction by finding a multicolored $P_5$.

We clearly have a path $P = v_1v_2v_3v_4$ colored with three distinct colors, say $v_iv_{i+1}$ colored with $i$. Both $v_1, v_4$ is incident to an edge of color 4, with other end on $P$. Suppose first that those edges coincide, i.e. $v_1v_4$ has color 4. Observe that no edge of color $i$ ($i = 1, 2, 3$) can go from an endpoint of an $i$-colored edge of $P$ to $V(G) \setminus V(P)$. Therefore - since colors 1 and 3 are connected - the pair of edges $v_1v_3, v_2v_4$ are colored with 1 and 3. But now the edge $v_2v_3$ is isolated in color 2 - contradiction.

Thus we may assume that edges $v_1v_3, v_2v_4$ are both colored with color 4. Now we get the same contradiction as before - the edge $v_2v_3$ is isolated in color 2. $\square$

**Problem 1.** Are paths in $GP$?

The next natural question is perhaps whether $C_4 \in GP$? This question have been asked some years ago by Simonyi and me, Ákos found a counterexample which will be presented here in a somewhat more general form. However, it might be true that $C_4 \in DP$.

**Problem 2.** $C_4 \in DP$?
The following result of Fujita and Magnant [2] provides an infinite family of graphs with GP.

**Theorem 2.** Suppose $F$ is a graph obtained from a star by adding a new edge between two endpoints. Then $F \in \mathcal{GP}$.

To get necessary conditions for $F \in \mathcal{GP}$ or $F \in \mathcal{DP}$ we define a specific $m$-colored complete graph, $K(m)$ for every $m \geq 2, k \geq 1$.

**Construction 1.** Let $A, B$ be disjoint sets, $|A| = |B| = 2(m - 1)k + 1$. Define

$$A = \bigcup_{i=1}^{m-1} A_i \cup \{a\}, B = \bigcup_{i=1}^{m-1} B_i \cup \{b\},$$

where the sets are all disjoint and $|A_i| = |B_i| = 2k$. Color 0 is distinguished, the edge $ab$ and the edges within $A$ and $B$ are colored with 0. For $i = 1, 2, \ldots, m - 1$, the edges of the complete bipartite graphs $[a, B_i], [b, A_i], [A_i, B_i]$ are colored with color $i$. Split each $A_i, B_i$ into two disjoint equal parts, $A_i = X_i \cup Y_i, B_i = U_i \cup W_i$ ($k$ vertices in each). For any $1 \leq i < j \leq m - 1$, color $[X_i, U_j], [Y_i, W_j]$ with color $i$ and color $[X_i, W_j], [Y_i, U_j]$ with color $j$. This colors all edges of the complete graph induced by $A \cup B$, we shall refer to it as $K(m)$.

A graph is called unicyclic if it has exactly one cycle, i.e. it has exactly one component that is not a tree and that component can be obtained by adding an edge to a tree.

**Lemma 1.** Suppose that $H$ is a multicolored connected bipartite subgraph of Construction 1. Then $H$ is either acyclic or unicyclic. An edge of $H$ with color 0 can not be on a cycle of $H$.

**Proof.** \(\Box\)

**Corollary 1.** Suppose that $F \in \mathcal{DP}$ is connected and bipartite. Then $F$ is either acyclic or unicyclic.

**Proof.** Suppose that $F \in \mathcal{DP}$ is bipartite. Consider the coloring of Construction 1 with $m \geq |E(F)|$. Since the coloring is connected we have a multicolored copy of $F$. By Lemma 1 the proof is finished. \(\Box\)

The next corollary shows that DP graphs are very close to acyclic graphs.

**Corollary 2.** Suppose that $F \in \mathcal{DP}$. Then for some $e \in E(F)$, the graph $F - e$ is bipartite.
Proof. Suppose that $F \in DP$. Consider the connected coloring of Construction 1 with $m \geq |E(F)|$. This coloring must contain a multicolored $F$. At most one edge of $F$ can be colored with color 0 and all other colors span bipartite graphs. Thus by the removal of at most one edge, $F$ becomes bipartite. $\Box$

Corollary 2 implies that every connected $F \in DP$ can be reduced to a connected bipartite graph, the core of $F$ by deleting at most one edge. However, one can say more about the core graphs, they have to be close to acyclic graphs.

Corollary 3. Every connected $F \in DP$ can be obtained from an acyclic graph by adding at most two edges.

Proposition 2. $C_{2i} \not\in GP$.

Proof. Consider Construction 1 with $m = 2i$. If $C_{2i} \in GP$ then the coloring of $K(m)$ must contain a multicolored $F = C_{2i}$. However, this is possible only if color 0 is used on $F$ but this contradicts Lemma 1. $\Box$

Problem 3. Are odd cycles in $GP$? Or at least in $DP$?

Theorem 3. Assume that $F$ is a unicyclic graph such that its cycle is a triangle. Then $F \in DP$.

Corollary 4. Acyclic graphs are in $DP$.

Problem 4. Is $DP \setminus GP \neq \emptyset$?

There are several possibilities to sharpen Theorem 3. ($\sim N$ is not precise, means ”about” $N$) For example:

1. Of course it is possible that $F \in GP$ (like in Theorem 2) For example the graph $F(1,1,1)$ obtained when each vertex of the triangle sends one edge out (six vertices) is in $GP$ - needed some hours of thinking and there are cases... Of course nice infinite families would be better - preferably with nice proof.

2. To improve the obvious bound ($\sim 2k$) when $F$ has $k$ vertices.

3. I looked briefly at the case when $F$ is a triangle plus edges going out from the triangle - $F(a,b,c)$. For $F(a,a,0)$ the obvious bound $\sim 4a$ went down to $\sim 3a$ by a nontrivial idea. Probably there are possibilities here...

NEW BRANCH

With Stanley we looked at the following variation: a balanced complete bipartite graph is colored (instead of a complete graph) and all color classes are connected... Now Ákos’s construction does not work - perhaps there is a rainbow $C_4$ in every connected coloring....
References


