Cycles in Hypergraphs

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5. $t$-tight Berge-cycles
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Notation

- $K_n^{(r)}$ is the complete $r$-uniform hypergraph on $n$ vertices.

- If $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ is an $r$-uniform hypergraph and $x_1, \ldots, x_{r-1} \in V(\mathcal{H})$, then

$$
\deg(x_1, \ldots, x_{r-1}) = |\{ e \in E(\mathcal{H}) \mid \{x_1, \ldots, x_{r-1}\} \subset e \}|.
$$

- Then the minimum degree in an $r$-uniform hypergraph $\mathcal{H}$:

$$
\delta(\mathcal{H}) = \min_{x_1, \ldots, x_{r-1}} \deg(x_1, \ldots, x_{r-1}).
$$
Loose cycles

There are several natural definitions for a hypergraph cycle. We survey these different cycle notions and some results available for them. The first one is the loose cycle. The definition is similar for $K_n^{(r)}$.

**Definition**

$C_m$ is a loose cycle in $K_n^{(3)}$, if it has vertices $\{v_1, \ldots, v_m\}$ and edges

$$\{(v_1, v_2, v_3), (v_3, v_4, v_5), (v_5, v_6, v_7), \ldots, (v_{m-1}, v_m, v_1)\}$$

(so in particular $m$ is even).
Density Results for Loose cycles

Theorem (Kühn, Osthus ’06)

If $\mathcal{H}$ is a 3-uniform hypergraph with $n \geq n_0$ vertices and

$$\delta(\mathcal{H}) \geq \frac{n}{4} + \epsilon n,$$

then $\mathcal{H}$ contains a loose Hamiltonian cycle.
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Theorem (Keevash, Kühn, Mycroft, Osthus ’08)

If $\mathcal{H}$ is an $r$-uniform hypergraph with $n \geq n_0(r)$ vertices and

$$\delta(\mathcal{H}) \geq \frac{n}{2(r-1)} + \epsilon n,$$

then $\mathcal{H}$ contains a loose Hamiltonian cycle.
Han and Schacht introduced a generalization of loose Hamiltonian cycles, \(l\)-Hamiltonian cycles where two consecutive edges intersect in exactly \(l\) vertices. They proved the following density result (also presented at this conference):

**Theorem (Han, Schacht ’08)**

If \(\mathcal{H}\) is an \(r\)-uniform hypergraph with \(n \geq n_0(r)\) vertices, \(l < r/2\) and 

\[
\delta(\mathcal{H}) \geq \frac{n}{2(r - l)} + \epsilon n,
\]

then \(\mathcal{H}\) contains a loose \(l\)-Hamiltonian cycle.
Theorem (Haxell, Łuczak, Peng, Rödl, Ruciński, Simonovits, Skokan ’06)

Every 2-coloring (of the edges) of $K_n^{(3)}$ with $n \geq n_0$ contains a monochromatic loose $C_m$ with $m \sim \frac{4}{5}n$. 
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A sharp version was obtained recently by Skokan and Thoma (presented at this conference).
We were able to generalize the asymptotic result for general $r$. 
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Theorem (Gyárfás, G.S., Szemerédi ’07)

Every 2-coloring (of the edges) of $K_n^{(r)}$ with $n \geq n_0(r)$ contains a monochromatic loose $C_m$ with $m \sim \frac{2r-2}{2r-1}n$. 
Our second cycle type is the tight cycle. The definition is similar for $K_n^{(r)}$.

**Definition**

$C_m$ is a **tight cycle** in $K_n^{(3)}$, if it has vertices $\{v_1, \ldots, v_m\}$ and edges

$$\{(v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_5), \ldots, (v_m, v_1, v_2)\}.$$ 

Thus every set of 3 consecutive vertices along the cycle forms an edge.
Density Results for Tight cycles

Theorem (Rödl, Ruciński, Szemerédi ’06)

If $\mathcal{H}$ is a 3-uniform hypergraph with $n \geq n_0$ vertices and

$$\delta(\mathcal{H}) \geq \frac{n}{2} + \epsilon n,$$

then $\mathcal{H}$ contains a tight Hamiltonian cycle.
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If $H$ is a 3-uniform hypergraph with $n \geq n_0$ vertices and

$$\delta(H) \geq \frac{n}{2} + \epsilon n,$$

then $H$ contains a tight Hamiltonian cycle.

Recently the same authors generalized this for general $r$. 
Density Results for Tight cycles

**Theorem (Rödl, Ruciński, Szemerédi ’06)**

*If $\mathcal{H}$ is a 3-uniform hypergraph with $n \geq n_0$ vertices and*

$$\delta(\mathcal{H}) \geq \frac{n}{2} + \epsilon n,$$

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Recently the same authors generalized this for general $r$.

**Theorem (Rödl, Ruciński, Szemerédi ’08)**

*If $\mathcal{H}$ is an $r$-uniform hypergraph with $n \geq n_0(r)$ vertices and*

$$\delta(\mathcal{H}) \geq \frac{n}{2} + \epsilon n,$$

*then $\mathcal{H}$ contains a tight Hamiltonian cycle.*
Theorem (Haxell, Łuczak, Peng, Rödl, Ruciński, Skokan ’08)

For the smallest integer \( N = N(m) \) for which every 2-coloring of \( K_N^{(3)} \) contains a monochromatic tight \( C_m \) we have \( N \sim \frac{4}{3} m \) if \( m \) is divisible by 3, and \( N \sim 2m \) otherwise.

All the above results use various forms of the Hypergraph Regularity Lemma.
Our next cycle type is the classical Berge-cycle.

**Definition**

\[ C_m = (v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_1) \] is a Berge-cycle in \( K_n^{(r)} \), if

- \( v_1, \ldots, v_m \) are all distinct vertices.
- \( e_1, \ldots, e_m \) are all distinct edges.
- \( v_k, v_{k+1} \in e_k \) for \( k = 1, \ldots, m \), where \( v_{m+1} = v_1 \).
Next we introduce a new cycle definition, the $t$-tight Berge-cycle (name suggested by Jenő Lehel).

**Definition**

$C_m = (v_1, v_2, \ldots, v_m)$ is a $t$-tight Berge-cycle in $K_n^{(r)}$, if for each set $(v_i, v_{i+1}, \ldots, v_{i+t-1})$ of $t$ consecutive vertices along the cycle (mod m), there is an edge $e_i$ containing it and these edges are all distinct.

Special cases: For $t = 2$ we get ordinary Berge-cycles and for $t = r$ we get the tight cycle.
Theorem (Gyárfás, Lehel, G.S., Schelp, JCTB ’08)

Every 2-coloring of $K^{(3)}_n$ with $n \geq 5$ contains a monochromatic Hamiltonian (2-tight) Berge-cycle.
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We conjecture that this is a very special case of the following more general phenomenon.
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Every 2-coloring of $K_n^{(3)}$ with $n \geq 5$ contains a monochromatic Hamiltonian (2-tight) Berge-cycle.

We conjecture that this is a very special case of the following more general phenomenon.

Conjecture

For any fixed $2 \leq c, t \leq r$ satisfying $c + t \leq r + 1$ and sufficiently large $n$, if we color the edges of $K_n^{(r)}$ with $c$ colors, then there is a monochromatic Hamiltonian $t$-tight Berge-cycle.

In the theorem above we have $r = 3, c = t = 2$. 
On the \((c + t)\)-conjecture

If true, the conjecture is best possible:

**Theorem (Dorbec, Gravier, G.S., JGT ’08)**

For any fixed \(2 \leq c, t \leq r\) satisfying \(c + t > r + 1\) and sufficiently large \(n\), there is a coloring of the edges of \(K^{(r)}_n\) with \(c\) colors, such that the longest monochromatic \(t\)-tight Berge-cycle has length at most \(\left\lceil \frac{t(c-1)n}{t(c-1)+1} \right\rceil\).

**Sketch of the proof:** Let \(A_1, \ldots, A_{c-1}\) be disjoint vertex sets of size \(\left\lfloor \frac{n}{t(c-1)+1} \right\rfloor\).

- Color 1: \(r\)-edges NOT containing a vertex from \(A_1\).
- Color 2: \(r\)-edges NOT containing a vertex from \(A_2\) and not in color 1, ...
- Color \(c-1\): \(r\)-edges NOT containing a vertex from \(A_{c-1}\) and not in color 1, \(\ldots, c - 2\).
- Color \(c\): \(r\)-edges containing a vertex from each \(A_i\).
Now the statement is trivial for colors 1, 2, \ldots, c - 1. In color c in any \( t \)-tight Berge-cycle from \( t \) consecutive vertices one has to come from \( A_1 \cup \ldots \cup A_{c-1} \), since \( t > r - c + 1 \). So the length is at most

\[
t(c - 1)\left\lfloor \frac{n}{t(c - 1) + 1} \right\rfloor.
\]

\( A_1 \quad \frac{A_2}{\ldots} \quad A_{c-1} \)

\( \leq r - c + 1 < t \)
On the \((c + t)\)-conjecture

**Sharp** results on the \((c + t)\)-conjecture, i.e. the conjecture is known to be true in these cases:

- \(r = 3, c = t = 2\) (Gyárfás, Lehel, G.S., Schelp, JCTB ’08)
- \(r = 4, c = 2, t = 3\) (Gyárfás, G.S., Szemerédi ’08)

“Almost” **sharp** results on the \((c + t)\)-conjecture:

- \(r = 4, c = 3, t = 2\) (Gyárfás, G.S., Szemerédi ’08) Under the assumptions there is a monochromatic \(t\)-tight Berge-cycle of length at least \(n - 10\).

**Asymptotic** results on the \((c + t)\)-conjecture:

- \(t = 2 (c \leq r - 1)\) (Gyárfás, G.S., Szemerédi ’07) Under the assumptions there is a monochromatic \(t\)-tight Berge-cycle of length at least \((1 - \epsilon)n\).
Sketch of the proof for \( r = 4, \ c = 2, \ t = 3 \): A 2-coloring \( c \) is given on the edges of \( \mathcal{K} = K^{(4)}_n \). \( c \) defines a 2-multicoloring on the complete 3-uniform shadow hypergraph \( \mathcal{T} \) by coloring a triple \( T \) with the colors of the edges of \( \mathcal{K} \) containing \( T \). We say that \( T \in \mathcal{T} \) is good in color \( i \) if \( T \) is contained in at least two edges of \( \mathcal{K} \) of color \( i \) \((i = 1, 2)\). Let \( G \) be the shadow graph of \( \mathcal{K} \). Then using a result of Bollobás and Gyárfás we get:

**Lemma**

*Every edge \( xy \in E(G) \) is in at least \( n - 4 \) good triples of the same color.*

This defines a 2-multicoloring \( c^* \) on the shadow graph \( G \) by coloring \( xy \in E(G) \) with the color of the (at least \( n - 4 \)) good triples containing \( xy \). Using a well-known result about the Ramsey number of even cycles there is a monochromatic even cycle \( C \) of length \( 2t \) where \( t \sim n/3 \). Then the idea is to splice in the vertices in \( V \setminus C \) into every second edge of \( C \).
However, in general we were able to obtain only the following weaker result, where essentially we replace the sum $c + t$ with the product $ct$.

**Theorem (Dorbec, Gravier, G.S., JGT '08)**

For any fixed $2 \leq c, t \leq r$ satisfying $ct + 1 \leq r$ and $n \geq 2(t + 1)rc^2$, if we color the edges of $K_n^{(r)}$ with $c$ colors, then there is a monochromatic Hamiltonian $t$-tight Berge-cycle.
Assume that \( c + t > r + 1 \), so there is no Hamiltonian cycle. What is the length of the longest cycle? An example:

**Theorem (Gyárfás, G.S., ’07)**

*Every 3-coloring of the edges of \( K^{(3)}_n \) with \( n \geq n_0 \) contains a monochromatic (2-tight) Berge-cycle \( C_m \) with \( m \sim \frac{4}{5}n \).*

Roughly this is what we get from the construction above.

All the papers can be downloaded from my homepage: http://web.cs.wpi.edu/~gsarkozy/