Cycles in Hypergraphs

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Outline of Topics

Notation

- $K_n^{(r)}$ is the complete r-uniform hypergraph on n vertices.
- If $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ is an r-uniform hypergraph and $x_1, \ldots, x_{r-1} \in V(\mathcal{H})$, then

$$deg(x_1, \ldots, x_{r-1}) = |\{e \in E(\mathcal{H}) \mid \{x_1, \ldots, x_{r-1}\} \subset e\}|.$$

- Then the minimum degree in an r-uniform hypergraph $\mathcal{H}$:

$$\delta(\mathcal{H}) = \min_{x_1, \ldots, x_{r-1}} deg(x_1, \ldots, x_{r-1}).$$

Loose cycles

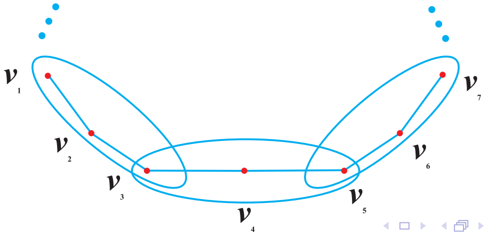
There are several natural definitions for a hypergraph cycle. We survey these different cycle notions and some results available for them. The first one is the loose cycle. The definition is similar for $K_n^{(r)}$.

Definition

C_m is a loose cycle in $K_n^{(3)}$, if it has vertices $\{v_1, \ldots, v_m\}$ and edges

$$\{(v_1, v_2, v_3), (v_3, v_4, v_5), (v_5, v_6, v_7), \ldots, (v_{m-1}, v_m, v_1)\}$$

(so in particular m is even).



Density Results for Loose cycles

Theorem (Kühn, Osthus '06)

If $\mathcal{H}$ is a 3-uniform hypergraph with $n \geq n_0$ vertices and

$$\delta(\mathcal{H}) \geq \frac{n}{4} + \epsilon n,$$

then $\mathcal{H}$ contains a loose Hamiltonian cycle.

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Theorem (Keevash, Kühn, Mycroft, Osthus '08)

If $\mathcal{H}$ is an r-uniform hypergraph with $n \geq n_0(r)$ vertices and

$$\delta(\mathcal{H}) \geq \frac{n}{2(r-1)} + \epsilon n,$$

then $\mathcal{H}$ contains a loose Hamiltonian cycle.

Density Results for Loose cycles

Han and Schacht introduced a generalization of loose Hamiltonian cycles, l-Hamiltonian cycles where two consecutive edges intersect in exactly l vertices. They proved the following density result (also presented at this conference):

Theorem (Han, Schacht '08)

If $\mathcal{H}$ is an r-uniform hypergraph with $n \geq n_0(r)$ vertices, $l < r/2$ and

$$\delta(\mathcal{H}) \geq \frac{n}{2(r-l)} + \epsilon n,$$

then $\mathcal{H}$ contains a loose l-Hamiltonian cycle.

Coloring Results for Loose cycles

Theorem (Haxell, Łuczak, Peng, Rödl, Ruciński, Simonovits, Skokan '06)

Every 2-coloring (of the edges) of $K_n^{(3)}$ *with* $n \geq n_0$ *contains a monochromatic loose* C_m *with* $m \sim \frac{4}{5}n$.

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Theorem (Gyárfás, G.S., Szemerédi '07)

Every 2-coloring (of the edges) of $K_n^{(r)}$ with $n \geq n_0(r)$ contains a monochromatic loose C_m with $m \sim \frac{2r-2}{2r-1}n$.

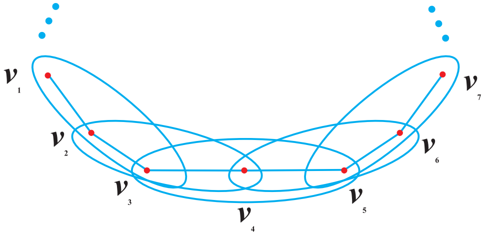
Tight cycles

Our second cycle type is the tight cycle. The definition is similar for $K_n^{(r)}$.

Definition

C_m is a tight cycle in $K_n^{(3)}$, if it has vertices $\{v_1, \ldots, v_m\}$ and edges

$$\{(v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_5), \ldots, (v_m, v_1, v_2)\}.$$



Thus every set of 3 consecutive vertices along the cycle forms an edge.

Density Results for Tight cycles

Theorem (Rödl, Ruciński, Szemerédi '06)

If $\mathcal{H}$ is a 3-uniform hypergraph with $n \geq n_0$ vertices and

$$\delta(\mathcal{H}) \geq \frac{n}{2} + \epsilon n,$$

then $\mathcal{H}$ contains a tight Hamiltonian cycle.

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Coloring Results for Tight cycles

Theorem (Haxell, Łuczak, Peng, Rödl, Ruciński, Skokan '08)

For the smallest integer $N = N(m)$ for which every 2-coloring of $K_N^{(3)}$ contains a monochromatic tight C_m we have $N \sim \frac{4}{3}m$ if m is divisible by 3, and $N \sim 2m$ otherwise.

All the above results use various forms of the Hypergraph Regularity Lemma.

Berge-cycles

Our next cycle type is the classical Berge-cycle.

Definition

$C_m = (v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_1)$ is a Berge-cycle in $K_n^{(r)}$, if

- $v_1, \ldots, v_m$ are all distinct vertices.
- $e_1, \ldots, e_m$ are all distinct edges.
- $v_k, v_{k+1} \in e_k$ for $k = 1, \ldots, m$, where $v_{m+1} = v_1$.

t-tight Berge-cycles

Next we introduce a new cycle definition, the t-tight Berge-cycle (name suggested by Jenő Lehel).

Definition

$C_m = (v_1, v_2, \ldots, v_m)$ is a t-tight Berge-cycle in $K_n^{(r)}$, if for each set $(v_i, v_{i+1}, \ldots, v_{i+t-1})$ of t consecutive vertices along the cycle (mod m), there is an edge e_i containing it and these edges are all distinct.

Special cases: For $t = 2$ we get ordinary Berge-cycles and for $t = r$ we get the tight cycle.

Coloring Results for t-Tight Berge-cycles

Theorem (Gyárfás, Lehel, G.S., Schelp, JCTB '08)

Every 2-coloring of $K_n^{(3)}$ *with* $n \geq 5$ *contains a monochromatic Hamiltonian (2-tight) Berge-cycle.*

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We conjecture that this is a very special case of the following more general phenomenon.

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We conjecture that this is a very special case of the following more general phenomenon.

Conjecture

For any fixed $2 \leq c, t \leq r$ satisfying $c + t \leq r + 1$ and sufficiently large n, if we color the edges of $K_n^{(r)}$ with c colors, then there is a monochromatic Hamiltonian t-tight Berge-cycle.

In the theorem above we have $r = 3$, $c = t = 2$.

On the $(c+t)$-conjecture

If true, the conjecture is best possible:

Theorem (Dorbec, Gravier, G.S., JGT '08)

For any fixed $2 \leq c, t \leq r$ satisfying $c + t > r + 1$ and sufficiently large n, there is a coloring of the edges of $K_n^{(r)}$ with c colors, such that the longest monochromatic t-tight Berge-cycle has length at most $\lceil \frac{t(c-1)n}{t(c-1)+1} \rceil$.

Sketch of the proof: Let $A_1, \ldots, A_{c-1}$ be disjoint vertex sets of size $\lfloor \frac{n}{t(c-1)+1} \rfloor$.

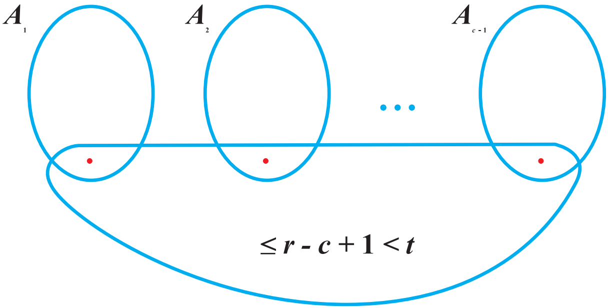
- Color 1: r-edges NOT containing a vertex from A_1.
- Color 2: r-edges NOT containing a vertex from A_2 and not in color 1,

  ...
- Color c-1: r-edges NOT containing a vertex from A_{c-1} and not in color $1, \ldots, c-2$.
- Color c: r-edges containing a vertex from each A_i.

On the $(c+t)$-conjecture

Now the statement is trivial for colors $1, 2, \ldots, c-1$. In color c in any t-tight Berge-cycle from t consecutive vertices one has to come from $A_1 \cup \ldots \cup A_{c-1}$, since $t > r - c + 1$. So the length is at most

$$t(c-1)\lfloor \frac{n}{t(c-1)+1} \rfloor.$$



On the $(c + t)$-conjecture

Sharp results on the $(c + t)$-conjecture, i.e. the conjecture is known to be true in these cases:

- $r = 3, c = t = 2$ (Gyárfás, Lehel, G.S., Schelp, JCTB '08)
- $r = 4$, $c = 2$, $t = 3$ (Gyárfás, G.S., Szemerédi '08)

"Almost" sharp results on the $(c + t)$-conjecture:

- $r = 4$, $c = 3$, $t = 2$ (Gyárfás, G.S., Szemerédi '08) Under the assumptions there is a monochromatic t-tight Berge-cycle of length at least $n - 10$.

Asymptotic results on the $(c + t)$-conjecture:

- $t = 2$ $(c \leq r - 1)$ (Gyárfás, G.S., Szemerédi '07) Under the assumptions there is a monochromatic t-tight Berge-cycle of length at least $(1 - \epsilon)n$.

On the $(c + t)$-conjecture

Sketch of the proof for $r = 4$, $c = 2$, $t = 3$: A 2-coloring c is given on the edges of $\mathcal{K} = K_n^{(4)}$. c defines a 2-multicoloring on the complete 3-uniform shadow hypergraph $\mathcal{T}$ by coloring a triple T with the colors of the edges of $\mathcal{K}$ containing T. We say that $T \in \mathcal{T}$ is *good in color i* if T is contained in at least two edges of $\mathcal{K}$ of color i $(i = 1, 2)$. Let G be the shadow graph of $\mathcal{K}$. Then using a result of Bollobás and Gyárfás we get:

Lemma

Every edge $xy \in E(G)$ is in at least $n - 4$ good triples of the same color.

This defines a 2-multicoloring c^* on the shadow graph G by coloring $xy \in E(G)$ with the color of the (at least $n - 4$) good triples containing xy. Using a well-known result about the Ramsey number of even cycles there is a monochromatic even cycle C of length $2t$ where $t \sim n/3$. Then the idea is to splice in the vertices in $V \setminus C$ into every second edge of C.

On the $(c+t)$-conjecture

However, in general we were able to obtain only the following weaker result, where essentially we replace the sum $c+t$ with the product ct.

Theorem (Dorbec, Gravier, G.S., JGT '08)

For any fixed $2 \leq c, t \leq r$ satisfying $ct+1 \leq r$ and $n \geq 2(t+1)rc^2$, if we color the edges of $K_n^{(r)}$ with c colors, then there is a monochromatic Hamiltonian t-tight Berge-cycle.

On the $(c + t)$-conjecture

Assume that $c + t > r + 1$, so there is no Hamiltonian cycle. What is the length of the longest cycle? An example:

Theorem (Gyárfás, G.S., '07)

Every 3-coloring of the edges of $K_n^{(3)}$ with $n \geq n_0$ contains a monochromatic (2-tight) Berge-cycle C_m with $m \sim \frac{4}{5}n$.

Roughly this is what we get from the construction above.

All the papers can be downloaded from my homepage:
http://web.cs.wpi.edu/∼gsarkozy/