On the Regularity Method

Gábor N. Sárközy

[1]Worcester Polytechnic Institute
USA

[2]Computer and Automation Research Institute
of the Hungarian Academy of Sciences
Budapest, Hungary

Co-authors: P. Dorbec, S. Gravier, A. Gyárfás, J. Lehel, R. Schelp and E. Szemerédi

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Introduction

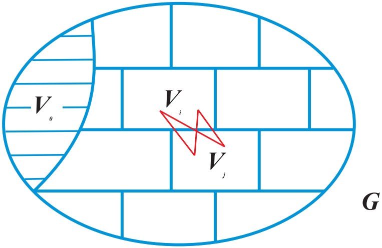
In many problems in graph (or hypergraph) theory we are faced with the following general problem: Given a dense graph G on a large number n of vertices (with $|E(G)| \geq c\binom{n}{2}$) we have to find a special (sometimes spanning) subgraph H in G. Typical examples for H include:

- Hamiltonian cycle or path
- Powers of a Hamiltonian cycle
- Coverings by special graphs
- Spanning subtrees, etc.

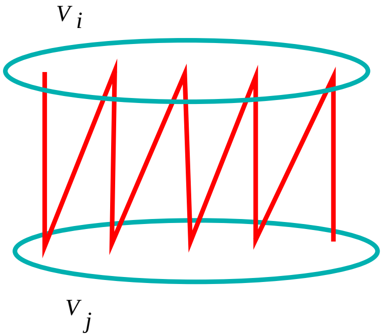
The Regularity method based on the Regularity Lemma (Szemerédi) and the Blow-up Lemma (Komlós, G.S., Szemerédi) works in these situations.

Introduction

Where do we start? We have to find some structure in G, the first step is to apply the Regularity Lemma for G. Roughly this says (details later) that apart from a small exceptional set V_0 we can partition the vertices into clusters $V_i, i \geq 1$ such that most of the pairs (V_i, V_j) are nice, random-looking (ϵ-regular).



Then we can "blow-up" a nice pair like this and the Blow-up Lemma claims that under some natural conditions any subgraph can be found in the pair. So roughly saying the Regularity Lemma finds the partition and then the Blow-up Lemma shows how to use this.



History of the Regularity method

- Regularity Lemma (Szemerédi '78)
- Weak hypergraph Regularity Lemma (Chung '91)
- Algorithmic version of the Regularity Lemma (Alon, Duke, Leffman, Rödl, Yuster '94)
- Blow-up Lemma (Komlós, G.S., Szemerédi '97)
- Algorithmic version of the Blow-up Lemma (Komlós, G.S., Szemerédi '98)
- Regularity method for graphs (Komlós, G.S., Szemerédi '96-...)
- Strong hypergraph Regularity Lemmas (Rödl, Nagle, Schacht, Skokan ' 04, Gowers '07, Tao '06, Elek, Szegedy '08, Ishigami '08)
- Hypergraph Blow-up Lemma (Keevash '08)
- Hypergraph Regularity method

Notation and definitions

- K_n is the complete graph on n vertices, $K(u, v)$ is the complete bipartite graph between U and V with $|U| = u, |V| = v$.
- $\delta(G)$ stands for the minimum, and $\Delta(G)$ for the maximum degree in G.
- When A, B are disjoint subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B. For non-empty A and B,

  $$d(A, B) = \frac{e(A, B)}{|A||B|}$$

  is the density of the graph between A and B.

Notation and definitions

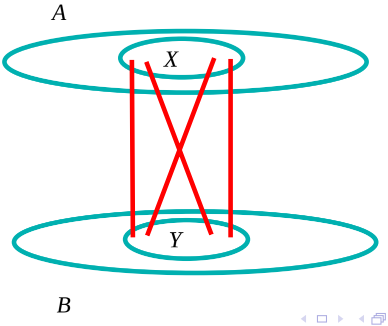
- The bipartite graph $G(A, B)$ (or simply the pair (A, B)) is called ϵ-regular if

$$X \subset A, \ Y \subset B, \ |X| > \epsilon|A|, \ |Y| > \epsilon|B|$$

imply

$$|d(X, Y) - d(A, B)| < \epsilon,$$

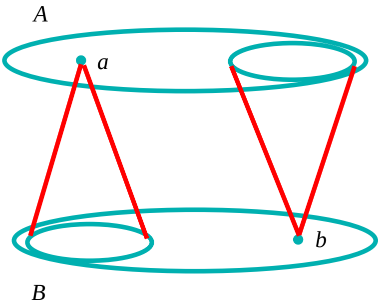
otherwise it is ϵ-irregular.



Notation and definitions

- (A, B) is (ϵ, δ)-super-regular if it is ϵ-regular and

$$deg(a) > \delta|B| \;\; \forall\, a \in A, \qquad deg(b) > \delta|A| \;\; \forall\, b \in B.$$



Regularity Lemma

Lemma (Regularity Lemma, Szemerédi '78)

For every $\epsilon > 0$ *and positive integer* m *there are positive integers* $M = M(\epsilon, m)$ *and* $N = N(\epsilon, m)$ *with the following property: for every graph* G *with at least* N *vertices there is a partition of the vertex set into* $l + 1$ *classes (clusters)*

$$V = V_0 + V_1 + V_2 + \ldots + V_l$$

such that

- $m \leq l \leq M$
- $|V_1| = |V_2| = \ldots = |V_l|$
- $|V_0| < \epsilon n$
- *apart from at most* $\epsilon\binom{l}{2}$ *exceptional pairs, all the pairs* (V_i, V_j) *are* ϵ*-regular.*

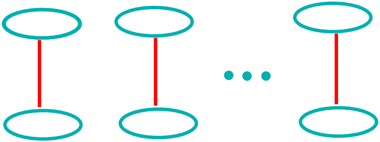
Overview of the Regularity method

So we have to find a special subgraph H in a dense graph G.
STEP 1: Preparation of G.
Decompose G into clusters by using the Regularity Lemma (with a small enough ϵ). Define the so-called reduced graph G_r: the vertices correspond to the clusters, $p_1, \ldots, p_l$, and we have an edge between p_i and p_j if the pair (V_i, V_j) is ϵ-regular with $d(V_i, V_j) \geq \delta$ (with some $\delta \gg \epsilon$). Then we have a one-to-one correspondence $f : p_i \rightarrow V_i$. Key observations:

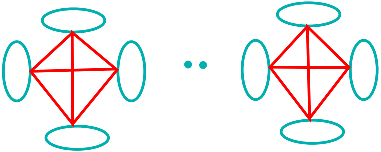
- G_r has only a constant number of vertices.
- G_r "inherits" the most important properties of G (e.g. degree and density conditions).
- G_r is the "essence" of G.

Overview of the Regularity method

STEP 2: Find "nice" objects in G_r.
This depends on the particular application and degree condition. Some examples:
Matching in G_r



Covering by cliques in G_r



Overview of the Regularity method

STEP 3: Preparation of H (if necessary).
STEP 4: "Technical manipulations".

- Connect the objects in the covering.
- Remove exceptional vertices from the clusters (just a few) to achieve super-regularity.
- Add the removed vertices to V_0.
- Redistribute the vertices of V_0 among the clusters in the covering.

The goal of STEP 4 is to reduce the embedding problem to embedding into the super-regular objects.

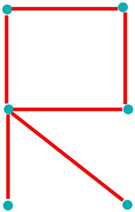
STEP 5: Finishing the embedding inside the super-regular objects.

Lemma (Blow-up Lemma, Komlós, G.S., Szemerédi '97)

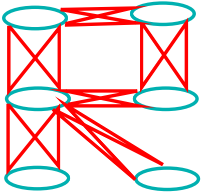
Given a graph R of order r and positive parameters δ, Δ, there exists an $\epsilon > 0$ such that the following holds. Let N be an arbitrary positive integer, and let us replace the vertices of R with pairwise disjoint N-sets $V_1, V_2, \ldots, V_r$ (blowing up). We construct two graphs on the same vertex-set $V = \cup V_i$. The graph $R(N)$ is obtained by replacing all edges of R with copies of the complete bipartite graph $K(N, N)$, and a sparser graph G is constructed by replacing the edges of R with some (ϵ, δ)-super-regular pairs. If a graph H with $\Delta(H) \leq \Delta$ is embeddable into $R(N)$ then it is already embeddable into G.

Overview of the Regularity method

We start from the graph R:

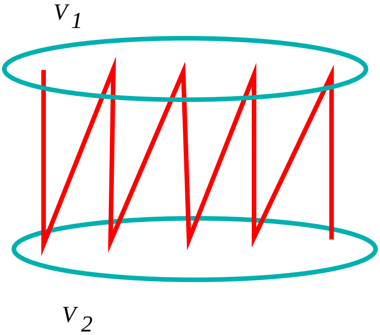


We blow it up and we have the graphs $H, G, R(N)$ on this new vertex set:



Overview of the Regularity method

Special case (R is just an edge): In a balanced (ϵ, δ)-super-regular pair G there is a Hamiltonian path H (max degree=2).



Remarks on the method:

- The method can be made algorithmic as both the Regularity Lemma and the Blow-up Lemma have algorithmic versions.
- The method only works for a really large $n \geq n_0$ (Gowers). In certain cases the method can be "de-regularized", i.e. the use of the Regularity Lemma can be avoided while maintaining some other key elements of the method. Then the resulting n_0 is much better.
- The method can be generalized for coloring problems. For this purpose we need an r-color version of the Regularity Lemma, we need a coloring in the reduced graph, etc.
- The method can be generalized for hypergraphs since by now the Hypergraph Regularity Lemma and the Hypergraph Blow-up Lemma are both available.

Some applications of the method

Proof of the Seymour conjecture for large graphs:

Theorem (Komlós, G.S., Szemerédi '98)

For any positive integer k there is an $n_0 = n_0(k)$ such that if G has order n with $n \geq n_0$ and $\delta(G) \geq \frac{k}{k+1}n$, then G contains the k^{th} power of a Hamiltonian cycle.

Proof of the Alon-Yuster conjecture for large graphs:

Theorem (Komlós, G.S., Szemerédi '01)

Let H be a graph with h vertices and chromatic number k. There exist constants $n_0(H), c(H)$ such that if $n \geq n_0(H)$ and G is a graph with hn vertices and minimum degree

$$\delta(G) \geq \left(1 - \frac{1}{k}\right) hn + c(H),$$

then G contains an H-factor.

Some applications of the method

Counting Hamiltonian cycles in Dirac graphs (a question of Bondy):

Theorem (G.S., Selkow, Szemerédi '03)

There exists a constant $c > 0$ such that the number of Hamiltonian cycles in Dirac graphs ($\delta(G) \geq n/2$) is at least $(cn)^n$.

This was recently improved by Cuckler and Kahn.
$R(G_1, G_2, \ldots, G_r)$ is the minimum n such that an arbitrary r-edge coloring of K_n contains a copy of G_i in color i for some i.
Proof of a conjecture of Faudree and Schelp for the 3-color Ramsey numbers for paths:

Theorem (Gyárfás, Ruszinkó, G.S., Szemerédi '07)

There exists an n_0 such that

$$R(P_n, P_n, P_n) = \begin{cases} 2n-1 & \text{for odd } n \geq n_0, \\ 2n-2 & \text{for even } n \geq n_0. \end{cases}$$

Additional notation for hypergraphs

- $K_n^{(r)}$ is the complete r-uniform hypergraph on n vertices.
- If $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ is an r-uniform hypergraph and $x_1, \ldots, x_{r-1} \in V(\mathcal{H})$, then

$$deg(x_1, \ldots, x_{r-1}) = \left| \{ e \in E(\mathcal{H}) \mid \{x_1, \ldots, x_{r-1}\} \subset e \} \right| .$$

- Then the minimum degree in an r-uniform hypergraph $\mathcal{H}$:

$$\delta(\mathcal{H}) = \min_{x_1, \ldots, x_{r-1}} deg(x_1, \ldots, x_{r-1}).$$

Loose cycles

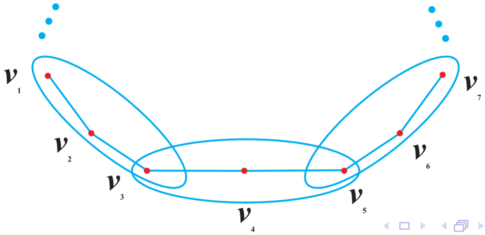
There are several natural definitions for a hypergraph cycle. We survey these different cycle notions and some results available for them. The first one is the loose cycle. The definition is similar for $K_n^{(r)}$.

Definition

C_m is a loose cycle in $K_n^{(3)}$, if it has vertices $\{v_1, \dots, v_m\}$ and edges

$$\{(v_1, v_2, v_3), (v_3, v_4, v_5), (v_5, v_6, v_7), \dots, (v_{m-1}, v_m, v_1)\}$$

(so in particular m is even).



Density Results for Loose cycles

Theorem (Kühn, Osthus '06)

If $\mathcal{H}$ is a 3-uniform hypergraph with $n \geq n_0$ vertices and

$$\delta(\mathcal{H}) \geq \frac{n}{4} + \epsilon n,$$

then $\mathcal{H}$ contains a loose Hamiltonian cycle.

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Theorem (Keevash, Kühn, Mycroft, Osthus '08)

If $\mathcal{H}$ is an r-uniform hypergraph with $n \geq n_0(r)$ vertices and

$$\delta(\mathcal{H}) \geq \frac{n}{2(r-1)} + \epsilon n,$$

then $\mathcal{H}$ contains a loose Hamiltonian cycle.

Density Results for Loose cycles

Han and Schacht introduced a generalization of loose Hamiltonian cycles, l-Hamiltonian cycles where two consecutive edges intersect in exactly l vertices. They proved the following density result:

Theorem (Han, Schacht '08)

If $\mathcal{H}$ is an r-uniform hypergraph with $n \geq n_0(r)$ vertices, $l < r/2$ and

$$\delta(\mathcal{H}) \geq \frac{n}{2(r-l)} + \epsilon n,$$

then $\mathcal{H}$ contains a loose l-Hamiltonian cycle.

Coloring Results for Loose cycles

Theorem (Haxell, Łuczak, Peng, Rödl, Ruciński, Simonovits, Skokan '06)

Every 2-coloring (of the edges) of $K_n^{(3)}$ with $n \geq n_0$ contains a monochromatic loose C_m with $m \sim \frac{4}{5}n$.

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Theorem (Gyárfás, G.S., Szemerédi EJC '08)

Every 2-coloring (of the edges) of $K_n^{(r)}$ with $n \geq n_0(r)$ contains a monochromatic loose C_m with $m \sim \frac{2r-2}{2r-1}n$.

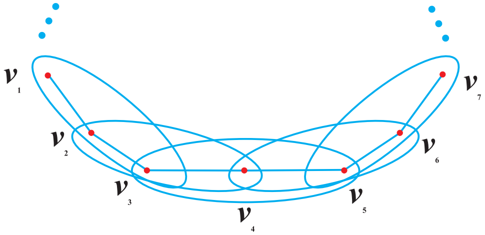
Tight cycles

Our second cycle type is the tight cycle. The definition is similar for $K_n^{(r)}$.

Definition

C_m is a tight cycle in $K_n^{(3)}$, if it has vertices $\{v_1, \ldots, v_m\}$ and edges

$$\{(v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_5), \ldots, (v_m, v_1, v_2)\}.$$



Thus every set of 3 consecutive vertices along the cycle forms an edge.

Density Results for Tight cycles

Improving a result of Katona and Kierstead:

Theorem (Rödl, Ruciński, Szemerédi '06)

If $\mathcal{H}$ is a 3-uniform hypergraph with $n \geq n_0$ vertices and

$$\delta(\mathcal{H}) \geq \frac{n}{2} + \epsilon n,$$

then $\mathcal{H}$ contains a tight Hamiltonian cycle.

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Coloring Results for Tight cycles

Theorem (Haxell, Łuczak, Peng, Rödl, Ruciński, Skokan '08)

For the smallest integer $N = N(m)$ for which every 2-coloring of $K_N^{(3)}$ contains a monochromatic tight C_m we have $N \sim \frac{4}{3}m$ if m is divisible by 3, and $N \sim 2m$ otherwise.

All the above cycle results use the hypergraph Regularity method.

Our next cycle type is the classical Berge-cycle.

Definition

$C_m = (v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_1)$ is a Berge-cycle in $K_n^{(r)}$, if

- $v_1, \ldots, v_m$ are all distinct vertices.
- $e_1, \ldots, e_m$ are all distinct edges.
- $v_k, v_{k+1} \in e_k$ for $k = 1, \ldots, m$, where $v_{m+1} = v_1$.

Next we introduce a new cycle definition, the t-tight Berge-cycle (name suggested by Jenő Lehel).

Definition

$C_m = (v_1, v_2, \ldots, v_m)$ is a t-tight Berge-cycle in $K_n^{(r)}$, if for each set $(v_i, v_{i+1}, \ldots, v_{i+t-1})$ of t consecutive vertices along the cycle (mod m), there is an edge e_i containing it and these edges are all distinct.

Special cases: For $t = 2$ we get ordinary Berge-cycles and for $t = r$ we get the tight cycle.

Coloring Results for t-Tight Berge-cycles

Theorem (Gyárfás, Lehel, G.S., Schelp, JCTB '08)

Every 2-coloring of $K_n^{(3)}$ with $n \geq 5$ contains a monochromatic Hamiltonian (2-tight) Berge-cycle.

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We conjecture that this is a very special case of the following more general phenomenon.

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Every 2-coloring of $K_n^{(3)}$ *with* $n \geq 5$ *contains a monochromatic Hamiltonian (2-tight) Berge-cycle.*

We conjecture that this is a very special case of the following more general phenomenon.

Conjecture (Dorbec, Gravier, G.S., JGT '08)

For any fixed $2 \leq c, t \leq r$ *satisfying* $c + t \leq r + 1$ *and sufficiently large* n*, if we color the edges of* $K_n^{(r)}$ *with* c *colors, then there is a monochromatic Hamiltonian* t*-tight Berge-cycle.*

In the theorem above we have $r = 3$, $c = t = 2$.

On the $(c+t)$-conjecture

If true, the conjecture is best possible:

Theorem (Dorbec, Gravier, G.S., JGT '08)

For any fixed $2 \le c, t \le r$ satisfying $c + t > r + 1$ and sufficiently large n, there is a coloring of the edges of $K_n^{(r)}$ with c colors, such that the longest monochromatic t-tight Berge-cycle has length at most $\lceil \frac{t(c-1)n}{t(c-1)+1} \rceil$.

Sketch of the proof: Let $A_1, \ldots, A_{c-1}$ be disjoint vertex sets of size $\lfloor \frac{n}{t(c-1)+1} \rfloor$.

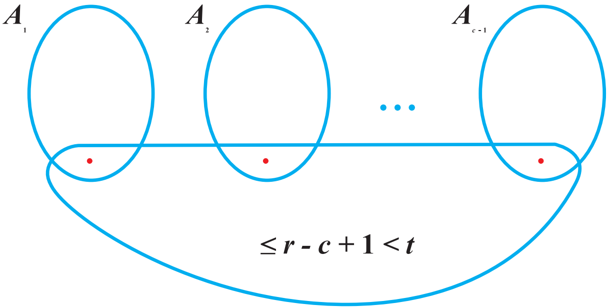
- Color 1: r-edges NOT containing a vertex from A_1.
- Color 2: r-edges NOT containing a vertex from A_2 and not in color 1,

  ...
- Color c-1: r-edges NOT containing a vertex from A_{c-1} and not in color $1, \ldots, c-2$.
- Color c: r-edges containing a vertex from each A_i.

On the $(c+t)$-conjecture

Now the statement is trivial for colors $1, 2, \ldots, c-1$. In color c in any t-tight Berge-cycle from t consecutive vertices one has to come from $A_1 \cup \ldots \cup A_{c-1}$, since $t > r - c + 1$. So the length is at most

$$t(c-1)\lfloor \frac{n}{t(c-1)+1} \rfloor.$$



On the $(c+t)$-conjecture

Sharp results on the $(c+t)$-conjecture, i.e. the conjecture is known to be true in these cases:

- $r = 3, c = t = 2$ (Gyárfás, Lehel, G.S., Schelp, JCTB '08)
- $r = 4,\ c = 2,\ t = 3$ (Gyárfás, G.S., Szemerédi '08)

"Almost" sharp results on the $(c+t)$-conjecture:

- $r = 4,\ c = 3,\ t = 2$ (Gyárfás, G.S., Szemerédi '08) Under the assumptions there is a monochromatic t-tight Berge-cycle of length at least $n - 10$.

Asymptotic results on the $(c+t)$-conjecture (using the Regularity method):

- $t = 2$ $(c \leq r - 1)$ (Gyárfás, G.S., Szemerédi '07) Under the assumptions there is a monochromatic t-tight Berge-cycle of length at least $(1-\epsilon)n$.

On the $(c+t)$-conjecture

Sketch of the proof for $r = 4$, $c = 2$, $t = 3$: A 2-coloring c is given on the edges of $\mathcal{K} = K_n^{(4)}$. c defines a 2-multicoloring on the complete 3-uniform shadow hypergraph $\mathcal{T}$ by coloring a triple T with the colors of the edges of $\mathcal{K}$ containing T. We say that $T \in \mathcal{T}$ is *good in color i* if T is contained in at least two edges of $\mathcal{K}$ of color i $(i = 1, 2)$. Let G be the shadow graph of $\mathcal{K}$. Then using a result of Bollobás and Gyárfás we get:

Lemma

Every edge $xy \in E(G)$ is in at least $n-4$ good triples of the same color.

This defines a 2-multicoloring c^* on the shadow graph G by coloring $xy \in E(G)$ with the color of the (at least $n-4$) good triples containing xy. Using a well-known result about the Ramsey number of even cycles there is a monochromatic even cycle C of length $2t$ where $t \sim n/3$. Then the idea is to splice in the vertices in $V \setminus C$ into every second edge of C.

On the $(c+t)$-conjecture

However, in general we were able to obtain only the following weaker result, where essentially we replace the sum $c+t$ with the product ct.

Theorem (Dorbec, Gravier, G.S., JGT '08)

For any fixed $2 \leq c, t \leq r$ satisfying $ct + 1 \leq r$ and $n \geq 2(t+1)rc^2$, if we color the edges of $K_n^{(r)}$ with c colors, then there is a monochromatic Hamiltonian t-tight Berge-cycle.

On the $(c + t)$-conjecture

Assume that $c + t > r + 1$, so there is no Hamiltonian cycle. What is the length of the longest cycle? An example:

Theorem (Gyárfás, G.S., '07)

Every 3-coloring of the edges of $K_n^{(3)}$ with $n \geq n_0$ contains a monochromatic (2-tight) Berge-cycle C_m with $m \sim \frac{4}{5}n$.

Roughly this is what we get from the construction above.

References

There are two excellent surveys on the topic:

- J. Komlós and M. Simonovits, "Szemerédi's Regularity Lemma and its applications in graph theory." in Combinatorics, Paul Erdős is Eighty (D. Miklós, V.T. Sós, and T. Szőnyi, Eds.), pp. 295-352, Bolyai Society Mathematical Studies, Vol. 2, János Bolyai Mathematical Society, Budapest, 1996.
- D. Kühn, D. Osthus, "Embedding large subgraphs into dense graphs." to appear.

All of my papers can be downloaded from my homepage:
http://web.cs.wpi.edu/∼gsarkozy/