Coverings by monochromatic pieces

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Outline of Topics

Introduction

Our main goal is to study the following problem:
General problem: Given fixed positive integers s, t, and a family of graphs $\mathcal{F}$, what is the maximum number of vertices that can be covered by the vertices of no more than s monochromatic members of $\mathcal{F}$ in every edge coloring of K_n with t colors? Let us introduce the notation $f(n, s, t, \mathcal{F})$ for this quantity. More precisely, $f(n, s, t, \mathcal{F})$ is the minimum (for all colorings) of the maximum size of all such covers.

Typical families $\mathcal{F}$: paths $\mathcal{P}$, cycles $\mathcal{C}$, matchings $\mathcal{M}$, connected matchings $\mathcal{CM}$ or simply connected components $\mathcal{CC}$.

This general problem unites two classical problems.

Introduction

- **One end of the spectrum:** $s = 1$, the Ramsey problem. Find the size of the largest monochromatic member of $\mathcal{F}$ that must be present in any edge coloring of a complete graph K_n with t colors. A difficult, classical problem, many papers.
- **The other end of the spectrum:** Cover problems (our main focus). We want to cover all the vertices by vertex disjoint monochromatic members of $\mathcal{F}$, how many do we need, i.e. for what value of s do we have $f(n, s, t, \mathcal{F}) = n$. Also a classical problem, for example an old Erdős-Gyárfás-Pyber conjecture states that $f(n, t, t, \mathcal{C}) = n$, i.e. we can always partition the vertex set into t monochromatic cycles.

But there are some interesting problems "in-between" as well.

Notation and definitions

- K_n is the complete graph on n vertices, $K(u,v)$ is the complete bipartite graph between U and V with $|U| = u, |V| = v$.
- $\delta(G)$ stands for the minimum degree, $\alpha(G)$ for the independence number of a graph G.
- When A, B are disjoint subsets of $V(G)$, we denote by $e(A,B)$ the number of edges of G with one endpoint in A and the other in B. For non-empty A and B,
$$d(A,B) = \frac{e(A,B)}{|A||B|}$$
is the density of the graph between A and B.

Notation and definitions

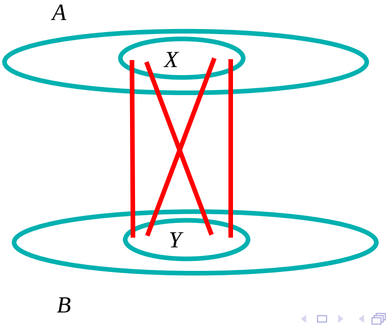
- The bipartite graph $G(A, B)$ (or simply the pair (A, B)) is called ϵ-regular if

$$X \subset A, \;\; Y \subset B, \;\; |X| > \epsilon|A|, \;\; |Y| > \epsilon|B|$$

imply

$$|d(X, Y) - d(A, B)| < \epsilon,$$

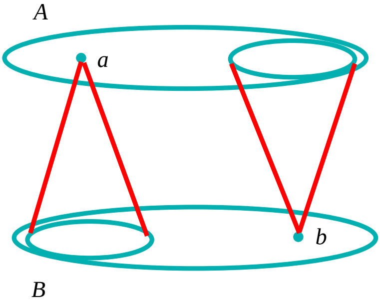
otherwise it is ϵ-irregular.



Notation and definitions

- (A, B) is (ϵ, δ)-super-regular if it is ϵ-regular and

$$deg(a) > \delta|B| \ \ \forall\ a \in A, \qquad deg(b) > \delta|A| \ \ \forall\ b \in B.$$



Overview of the Regularity method

Our main proof method is the Regularity Method based on the Regularity Lemma (Szemerédi '78) and the Blow-up Lemma (Komlós, G.S., Szemerédi '97), so before we get into the results we will give a quick review of this method. Here the Regularity Lemma finds an ϵ-regular partition and the Blow-up Lemma shows how to use this.

Regularity Lemma

Lemma (Regularity Lemma, Szemerédi '78)

For every $\epsilon > 0$ *and positive integer* m *there are positive integers* $M = M(\epsilon, m)$ *and* $N = N(\epsilon, m)$ *with the following property: for every graph* G *with at least* N *vertices there is a partition of the vertex set into* $l + 1$ *classes (clusters)*

$$V = V_0 + V_1 + V_2 + \ldots + V_l$$

such that

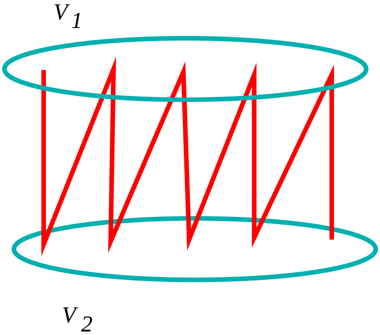
- $m \leq l \leq M$
- $|V_1| = |V_2| = \ldots = |V_l|$
- $|V_0| < \epsilon n$
- *apart from at most* $\epsilon\binom{l}{2}$ *exceptional pairs, all the pairs* (V_i, V_j) *are* ϵ*-regular.*

Overview of the Regularity method

Decompose G into clusters by using the Regularity Lemma (with a small enough ϵ). Define the so-called reduced graph G_r: the vertices correspond to the clusters, $p_1, \ldots, p_l$, and we have an edge between p_i and p_j if the pair (V_i, V_j) is ϵ-regular with $d(V_i, V_j) \geq \delta$ (with some $\delta \gg \epsilon$). Then we have a one-to-one correspondence $f : p_i \rightarrow V_i$. Key observations:

- G_r has only a constant number of vertices.
- G_r "inherits" the most important properties of G (e.g. degree and density conditions).
- G_r is the "essence" of G.
- If G is colored then we can define a coloring in G_r as well.

Special case of the Blow-up Lemma: In a balanced (ϵ, δ)-super-regular pair G there is a Hamiltonian path H (max degree=2).



Using this we can get our main tool:
If we have a connected matching in G_r, then we can span most of the vertices in these clusters by a path or cycle in G, i.e. we can "lift" the connected matching back into a path or cycle in the original graph. Thus roughly speaking

$$f(n, s, t, \mathcal{P}) \sim f(n, s, t, \mathcal{CM}).$$

(An idea first observed by Łuczak.)

One end of the spectrum: the Ramsey problem

Recall the definition of $f(n, s, t, \mathcal{F})$.
Here we have $s = 1$. We consider paths $\mathcal{P}$.
For $t = 2$ we have

$$f(n, 1, 2, \mathcal{P}) \sim \frac{2n}{3}.$$

More precisely, using the inverse Ramsey formulation:

Theorem (Gerencsér, Gyárfás '67)

$$R(P_n, P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor.$$

The Ramsey problem

For $t = 3$ we have

$$f(n, 1, 3, \mathcal{P}) \sim \frac{n}{2}.$$

More precisely (for large n):

Theorem (Gyárfás, Ruszinkó, G.S., Szemerédi '07)

There exists an n_0 such that

$$R(P_n, P_n, P_n) = \begin{cases} 2n - 1 & \text{for odd } n \geq n_0, \\ 2n - 2 & \text{for even } n \geq n_0. \end{cases}$$

Proof ideas: Regularity method +

$$f(n, 1, 3, \mathcal{P}) \sim f(n, 1, 3, \mathcal{CM}) \sim f(n, 1, 3, \mathcal{M}) \sim f(n, 1, 3, \mathcal{CC}) \sim \frac{n}{2}.$$

The Ramsey problem

Recently we extended this (at least asymptotically) for the following larger family of graphs:

Definition

A bipartite graph H is called a (β, Δ)-graph if it has bandwidth at most $\beta|V(H)|$ and maximum degree at most Δ. Furthermore, we say that H is a balanced (β, Δ)-graph if it has a legal 2-coloring $\chi : V(H) \to [2]$ such that $1 - \beta \leq |\chi^{-1}(1)|/|\chi^{-1}(2)| \leq 1 + \beta$.

Theorem (Mota, G.S., Schacht, Taraz '13)

For every $\gamma > 0$ and natural number Δ, there exist a constant $\beta > 0$ and natural number n_0 such that for every balanced (β, Δ)-graph H on $n \geq n_0$ vertices we have

$$R(H, H, H) \leq (2 + \gamma)n.$$

The Ramsey problem

Going back to paths what about $t = 4$ (or higher)? Wide open. The above is not true anymore:

$$f(n, 1, 4, \mathcal{M}) \sim \frac{2n}{5}, f(n, 1, 4, \mathcal{CC}) \sim \frac{n}{3}.$$

We believe:

$$f(n, 1, 4, \mathcal{P}) \sim f(n, 1, 4, \mathcal{CM}) \sim f(n, 1, 4, \mathcal{CC}) \sim \frac{n}{3}.$$

The other end of the spectrum: cover problems

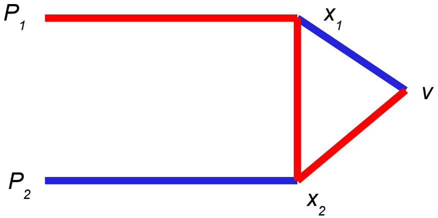
Here we want $f(n, s, t, \mathcal{F}) = n$.
First $t = 2$ and $\mathcal{F} = \mathcal{P}$:

Claim

$$f(n, 2, 2, \mathcal{P}) = n,$$

in fact we can partition into 2 monochromatic paths of different color.

Proof: Either v can be placed to the end of P_1 or P_2 or (x_1, v) is blue and (x_2, v) is red. Then let's look at (x_1, x_2), wlog it's red, then we can extend P_1 by x_2, v.



Cover problems

Next $t = 2$ and $\mathcal{F} = \mathcal{C}$. Lehel conjectured that the same is true for cycles:

$$f(n, 2, 2, \mathcal{C}) = n,$$

where again we can partition into 2 monochromatic cycles of different color.

- Łuczak, Rödl, Szemerédi '98: proof for $n \geq n_0$ (using the Regularity Method).
- Allen '08: improved on n_0.
- Bessy, Thomassé '09: for all n.

For general t Erdős-Gyárfás-Pyber conjecture:

Conjecture

$$f(n, t, t, \mathcal{C}) = n.$$

(Here single vertices, edges and the empty set are considered to be degenerate cycles). This would be best possible, we need at least t cycles.

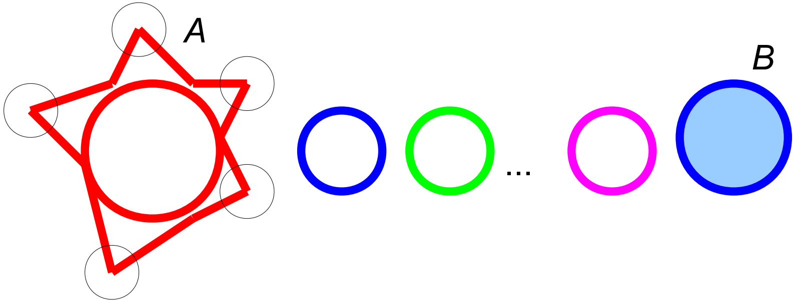
Theorem (Erdős, Gyárfás, Pyber '91)

We can cover by $\leq ct^2 \log t$ *vertex disjoint monochromatic cycles.*

Proof sketch: (Absorbing method.)

- **Step 1:** Find a large monochromatic (say red) triangle cycle. Property: If A is the set of "third" vertices in the triangles, then if we remove a subset of A there is still a spanning red cycle.
- **Step 2:** Greedily remove monochromatic cycles until the leftover B is small compared to A.
- **Step 3:** Unbalanced bipartite cover lemma between A and B. (The triangle cycle absorbs the leftover.)

Cover problems



Cover problems

Current best result for general t:

Theorem (Gyárfás, Ruszinkó, G.S., Szemerédi '06)

For every integer $t \geq 2$ there exists a constant $n_0 = n_0(t)$ such that if $n \geq n_0$ and the edges of the complete graph K_n are colored with t colors then the vertex set of K_n can be partitioned into at most $100t \log t$ vertex disjoint monochromatic cycles.

Proof idea: Regularity Method combined with the absorbing method, the triangle cycle is replaced with a larger monochromatic absorbing structure, a dense, connected matching. However, the greedy procedure stays, that's why we have the $\log t$.

Cover problems

$t = 3$:

- Gyárfás, Ruszinkó, G.S., Szemerédi '11: $\geq (1 - \epsilon)n$ vertices can be covered by 3 monochromatic cycles.
- n vertices can be covered by 3 connected matchings.
- n vertices can be covered by 17 monochromatic cycles.

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- n vertices can be covered by 3 connected matchings.
- n vertices can be covered by 17 monochromatic cycles.
- Pokrovskiy '12: n vertices can be covered by 3 monochromatic paths.
- Pokrovskiy '12: The conjecture is not true for any $t \geq 3$.

However, in the counterexample all but one vertex can be covered by t vertex disjoint monochromatic cycles. So perhaps the following weaker conjecture is true:

Conjecture

Let G be a t-colored graph. Then there exist a constant $c = c(t)$ and t vertex disjoint monochromatic cycles of G that cover at least $n - c$ vertices.

Generalized cover problems

1st generalization: non-complete graphs, we t-color a graph G with $\alpha(G) = \alpha$. We may define $f(n, \alpha, s, t, \mathcal{F})$ in a similar way.

Conjecture (G.S. '11)

$$f(n, \alpha, t\alpha, t, \mathcal{C}) = n.$$

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For $t = 1$, this is a well-known result of Pósa (and clearly best possible).

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For $t = 1$, this is a well-known result of Pósa (and clearly best possible). For $t = 2$ it would also be best possible. However, we only have an asymptotic result:

Theorem (Balog, Barát, Gerbner, Gyárfás, G.S. '12)

For every positive η and α, there exists an $n_0(\eta, \alpha)$ such that the following holds. If G is a 2-colored graph on n vertices, $n \geq n_0$, $\alpha(G) = \alpha$, then there are at most 2α vertex disjoint monochromatic cycles covering at least $(1 - \eta)n$ vertices of $V(G)$.

Generalized cover problems

For a general t we have the following result:

Theorem (G.S. '11)

The vertex set of any t-colored G with $\alpha(G) = \alpha$ can be partitioned into at most $25(\alpha t)^2 \log(\alpha t)$ vertex disjoint monochromatic cycles.

Proof idea: Absorbing Method + induction on α.

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Proof idea: Absorbing Method + induction on α.
Unfortunately, Pokrovskiy's counterexample disproves this conjecture as well. Perhaps the following weaker conjecture is true:

Conjecture

Let G be a t-colored graph with $\alpha(G) = \alpha$. Then there exist a constant $c = c(\alpha, t)$ and $t\alpha$ vertex disjoint monochromatic cycles of G that cover at least $n - c$ vertices.

Pokrovskiy's counterexample implies that $c \geq \alpha$.

Generalized cover problems

2nd generalization: non-complete graphs, we t-color a graph G with $\delta(G) > \delta$. We may define $f(n, \delta, s, t, \mathcal{F})$ in a similar way.

Conjecture

$$f(n, \frac{3n}{4}, 2, 2, \mathcal{C}) = n,$$

where again we can partition into 2 monochromatic cycles of different color.

Thus the Bessy-Thomassé result would hold for graphs with minimum degree larger than $3n/4$ (sharp). Again, we only have an asymptotic result:

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Thus the Bessy-Thomassé result would hold for graphs with minimum degree larger than $3n/4$ (sharp). Again, we only have an asymptotic result:

Theorem (Balog, Barát, Gerbner, Gyárfás, G.S. '12)

For every $\eta > 0$, there is an $n_0(\eta)$ such that if G is a graph on $n \geq n_0$ vertices, $\delta(G) > (\frac{3}{4} + \eta)n$, then every 2-edge-coloring of G admits two vertex disjoint monochromatic cycles of different colors covering at least $(1 - \eta)n$ vertices of G.

Generalized cover problems

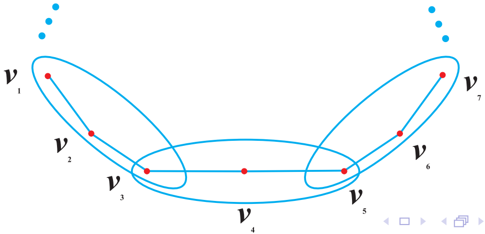
3rd generalization: hypergraphs, we t-color the edges of the complete k-uniform hypergraph $K_n^{(k)}$. We may define $f_k(n, s, t, \mathcal{F})$ in a similar way. Let us consider loose cycles first. The definition is similar for $K_n^{(k)}$.

Definition

C_m is a loose cycle in $K_n^{(3)}$, if it has vertices $\{v_1, \ldots, v_m\}$ and edges

$$\{(v_1, v_2, v_3), (v_3, v_4, v_5), (v_5, v_6, v_7), \ldots, (v_{m-1}, v_m, v_1)\}$$

(so in particular m is even).



We have the following result for loose cycles (improving an earlier result):

Theorem (G.S. '12)

For all integers $t, k \geq 2$ there exists a constant $n_0 = n_0(t, k)$ such that if $n \geq n_0$ and the edges of the complete k-uniform hypergraph $K_n^{(k)}$ are colored with t colors then the vertex set can be partitioned into at most $50tk \log(tk)$ vertex disjoint monochromatic loose cycles.

The proof is using the Strong Hypergraph Regularity Lemma and the recent Hypergraph Blow-up Lemma of Keevash.
We do not risk an exact conjecture here. It would be nice to prove a similar result for tight cycles.

In-between problems

Returning to the original $f(n, s, t, \mathcal{P})$. Many open problems. Let us mention one interesting problem here:

Conjecture

$$f(n, 2, 3, \mathcal{P}) \sim f(n, 2, 3, \mathcal{C}) \sim \frac{6n}{7}.$$

The reason why we believe this is the following theorem:

Theorem (Gyárfás, G.S., Selkow '11)

$$f(n, t-1, t, \mathcal{M}) \sim \frac{(2^t - 2)n}{2^t - 1}, \text{ so } f(n, 2, 3, \mathcal{M}) \sim \frac{6n}{7}.$$

If we could generalize this for $\mathcal{CM}$, then we would get the conjecture.

References

Most of the problems and results mentioned can be found in:

- G.N. Sárközy, "Coverings by monochromatic pieces - problems for the Emléktábla workshop." Proceedings of the 3rd Emléktábla Workshop, János Bolyai Mathematical Society, 2011, pp. 1-9.

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Thank you!