Coverings by monochromatic pieces

Gábor N. Sárközy

1Worcester Polytechnic Institute
USA

2Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, Hungary

March 17, 2013
Outline of Topics

1. Introduction: the general problem
2. Notation and definitions
3. Overview of the Regularity method
4. One end of the spectrum: the Ramsey problem
5. The other end of the spectrum: cover problems
6. Generalized cover problems
7. In-between problems
Our main goal is to study the following problem:

**General problem:** Given fixed positive integers $s$, $t$, and a family of graphs $\mathcal{F}$, what is the maximum number of vertices that can be covered by the vertices of no more than $s$ monochromatic members of $\mathcal{F}$ in every edge coloring of $K_n$ with $t$ colors? Let us introduce the notation $f(n, s, t, \mathcal{F})$ for this quantity. More precisely, $f(n, s, t, \mathcal{F})$ is the minimum (for all colorings) of the maximum size of all such covers.

Typical families $\mathcal{F}$: paths $\mathcal{P}$, cycles $\mathcal{C}$, matchings $\mathcal{M}$, connected matchings $\mathcal{CM}$ or simply connected components $\mathcal{CC}$.

This general problem unites two classical problems.
• One end of the spectrum: $s = 1$, the Ramsey problem. Find the size of the largest monochromatic member of $\mathcal{F}$ that must be present in any edge coloring of a complete graph $K_n$ with $t$ colors. A difficult, classical problem, many papers.

• The other end of the spectrum: Cover problems (our main focus). We want to cover all the vertices by vertex disjoint monochromatic members of $\mathcal{F}$, how many do we need, i.e. for what value of $s$ do we have $f(n, s, t, \mathcal{F}) = n$. Also a classical problem, for example an old Erdős-Gyárfás-Pyber conjecture states that $f(n, t, t, \mathcal{C}) = n$, i.e. we can always partition the vertex set into $t$ monochromatic cycles.

But there are some interesting problems “in-between” as well.
Notation and definitions

- $K_n$ is the complete graph on $n$ vertices, $K(u, v)$ is the complete bipartite graph between $U$ and $V$ with $|U| = u, |V| = v$.
- $\delta(G)$ stands for the minimum degree, $\alpha(G)$ for the independence number of a graph $G$.
- When $A, B$ are disjoint subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of $G$ with one endpoint in $A$ and the other in $B$. For non-empty $A$ and $B$,
  \[ d(A, B) = \frac{e(A, B)}{|A||B|} \]
  is the density of the graph between $A$ and $B$. 

Sárközy (WPI–Renyi)  Coverings by monochromatic pieces  March 17, 2013  5 / 30
The bipartite graph $G(A, B)$ (or simply the pair $(A, B)$) is called $\epsilon$-regular if

$$X \subset A, \ Y \subset B, \ |X| > \epsilon|A|, \ |Y| > \epsilon|B|$$

imply

$$|d(X, Y) - d(A, B)| < \epsilon,$$

otherwise it is $\epsilon$-irregular.
(A, B) is \((\epsilon, \delta)\)-super-regular if it is \(\epsilon\)-regular and

\[
\deg(a) > \delta |B| \quad \forall \ a \in A, \quad \deg(b) > \delta |A| \quad \forall \ b \in B.
\]
Our main proof method is the Regularity Method based on the Regularity Lemma (Szemerédi ’78) and the Blow-up Lemma (Komlós, G.S., Szemerédi ’97), so before we get into the results we will give a quick review of this method. Here the Regularity Lemma finds an $\epsilon$-regular partition and the Blow-up Lemma shows how to use this.
Lemma (Regularity Lemma, Szemerédi ’78)

For every $\epsilon > 0$ and positive integer $m$ there are positive integers $M = M(\epsilon, m)$ and $N = N(\epsilon, m)$ with the following property: for every graph $G$ with at least $N$ vertices there is a partition of the vertex set into $l + 1$ classes (clusters)

$$V = V_0 + V_1 + V_2 + \ldots + V_l$$

such that

- $m \leq l \leq M$
- $|V_1| = |V_2| = \ldots = |V_l|$
- $|V_0| < \epsilon n$
- apart from at most $\epsilon \binom{l}{2}$ exceptional pairs, all the pairs $(V_i, V_j)$ are $\epsilon$-regular.
Overview of the Regularity method

Decompose $G$ into clusters by using the Regularity Lemma (with a small enough $\epsilon$). Define the so-called reduced graph $G_r$: the vertices correspond to the clusters, $p_1, \ldots, p_l$, and we have an edge between $p_i$ and $p_j$ if the pair $(V_i, V_j)$ is $\epsilon$-regular with $d(V_i, V_j) \geq \delta$ (with some $\delta \gg \epsilon$). Then we have a one-to-one correspondence $f : p_i \rightarrow V_i$. Key observations:

- $G_r$ has only a constant number of vertices.
- $G_r$ “inherits” the most important properties of $G$ (e.g. degree and density conditions).
- $G_r$ is the “essence” of $G$.
- If $G$ is colored then we can define a coloring in $G_r$ as well.
Special case of the Blow-up Lemma: In a balanced \((\epsilon, \delta)\)-super-regular pair \(G\) there is a Hamiltonian path \(H\) (max degree=2).
Using this we can get our main tool:
If we have a connected matching in $G_r$, then we can span most of the vertices in these clusters by a path or cycle in $G$, i.e. we can “lift” the connected matching back into a path or cycle in the original graph. Thus roughly speaking

$$f(n, s, t, \mathcal{P}) \sim f(n, s, t, \mathcal{CM}).$$

(An idea first observed by Łuczak.)
One end of the spectrum: the Ramsey problem

Recall the definition of $f(n, s, t, \mathcal{F})$. Here we have $s = 1$. We consider paths $\mathcal{P}$. For $t = 2$ we have

$$f(n, 1, 2, \mathcal{P}) \sim \frac{2n}{3}. $$

More precisely, using the inverse Ramsey formulation:

**Theorem (Gerencsér, Gyárfás ’67)**

$$R(P_n, P_n) = \left\lfloor \frac{3n - 2}{2} \right\rfloor. $$
The Ramsey problem

For \( t = 3 \) we have

\[
f(n, 1, 3, \mathcal{P}) \sim \frac{n}{2}.
\]

More precisely (for large \( n \)):

**Theorem (Gyárfás, Ruszinkó, G.S., Szemerédi '07)**

There exists an \( n_0 \) such that

\[
R(P_n, P_n, P_n) = \begin{cases} 
2n - 1 & \text{for odd } n \geq n_0, \\
2n - 2 & \text{for even } n \geq n_0.
\end{cases}
\]

Proof ideas: Regularity method +

\[
f(n, 1, 3, \mathcal{P}) \sim f(n, 1, 3, \mathcal{CM}) \sim f(n, 1, 3, \mathcal{M}) \sim f(n, 1, 3, \mathcal{CC}) \sim \frac{n}{2}.
\]
The Ramsey problem

Recently we extended this (at least asymptotically) for the following larger family of graphs:

**Definition**

A bipartite graph $H$ is called a $(\beta, \Delta)$-graph if it has bandwidth at most $\beta|V(H)|$ and maximum degree at most $\Delta$. Furthermore, we say that $H$ is a balanced $(\beta, \Delta)$-graph if it has a legal 2-coloring $\chi : V(H) \rightarrow [2]$ such that $1 - \beta \leq |\chi^{-1}(1)|/|\chi^{-1}(2)| \leq 1 + \beta$.

**Theorem (Mota, G.S., Schacht, Taraz '13)**

For every $\gamma > 0$ and natural number $\Delta$, there exist a constant $\beta > 0$ and natural number $n_0$ such that for every balanced $(\beta, \Delta)$-graph $H$ on $n \geq n_0$ vertices we have

$$R(H, H, H) \leq (2 + \gamma)n.$$
Going back to paths what about $t = 4$ (or higher)? Wide open. The above is not true anymore:

$$f(n, 1, 4, M) \sim \frac{2n}{5}, f(n, 1, 4, CC) \sim \frac{n}{3}.$$ 

We believe:

$$f(n, 1, 4, P) \sim f(n, 1, 4, CM) \sim f(n, 1, 4, CC) \sim \frac{n}{3}.$$
Here we want $f(n, s, t, \mathcal{F}) = n$.
First $t = 2$ and $\mathcal{F} = \mathcal{P}$:

Claim

$$f(n, 2, 2, \mathcal{P}) = n,$$

in fact we can partition into 2 monochromatic paths of different color.

Proof: Either $v$ can be placed to the end of $P_1$ or $P_2$ or $(x_1, v)$ is blue and $(x_2, v)$ is red. Then let’s look at $(x_1, x_2)$, wlog it’s red, then we can extend $P_1$ by $x_2, v$. 

![Diagram of two paths and vertices](image-url)
Next $t = 2$ and $\mathcal{F} = \mathcal{C}$. Lehel conjectured that the same is true for cycles:

$$f(n, 2, 2, \mathcal{C}) = n,$$

where again we can partition into 2 monochromatic cycles of different color.

- Łuczak, Rödl, Szemerédi ’98: proof for $n \geq n_0$ (using the Regularity Method).
- Allen ’08: improved on $n_0$.
- Bessy, Thomassé ’09: for all $n$. 
Cover problems

For general $t$ Erdős-Gyárfás-Pyber conjecture:

**Conjecture**

$$f(n, t, t, C) = n.$$  

(Here single vertices, edges and the empty set are considered to be degenerate cycles). This would be best possible, we need at least $t$ cycles.

**Theorem (Erdős, Gyárfás, Pyber ’91)**

*We can cover by $\leq ct^2 \log t$ vertex disjoint monochromatic cycles.*
Proof sketch: (Absorbing method.)

- **Step 1:** Find a large monochromatic (say red) triangle cycle. Property: If $A$ is the set of "third" vertices in the triangles, then if we remove a subset of $A$ there is still a spanning red cycle.

- **Step 2:** Greedily remove monochromatic cycles until the leftover $B$ is small compared to $A$.

- **Step 3:** Unbalanced bipartite cover lemma between $A$ and $B$. (The triangle cycle absorbs the leftover.)
Cover problems

A

B

...
Current best result for general $t$:

**Theorem (Gyárfás, Ruszinkó, G.S., Szemerédi ’06)**

For every integer $t \geq 2$ there exists a constant $n_0 = n_0(t)$ such that if $n \geq n_0$ and the edges of the complete graph $K_n$ are colored with $t$ colors then the vertex set of $K_n$ can be partitioned into at most $100t \log t$ vertex disjoint monochromatic cycles.

Proof idea: Regularity Method combined with the absorbing method, the triangle cycle is replaced with a larger monochromatic absorbing structure, a dense, connected matching. However, the greedy procedure stays, that’s why we have the log $t$. 
Cover problems

$t = 3$:

- Gyárfás, Ruszinkó, G.S., Szemerédi '11: $\geq (1 - \epsilon)n$ vertices can be covered by 3 monochromatic cycles.
- $n$ vertices can be covered by 3 connected matchings.
- $n$ vertices can be covered by 17 monochromatic cycles.

Pokrovskiy '12: The conjecture is not true for any $t \geq 3$. However, in the counterexample all but one vertex can be covered by $t$ vertex disjoint monochromatic cycles. So perhaps the following weaker conjecture is true:

**Conjecture**

Let $G$ be a $t$-colored graph. Then there exist a constant $c = c(t)$ and $t$ vertex disjoint monochromatic cycles of $G$ that cover at least $n - c$ vertices.
Cover problems

$t = 3$:

- Gyárfás, Ruszinkó, G.S., Szemerédi '11: $\geq (1 - \epsilon)n$ vertices can be covered by 3 monochromatic cycles.
- $n$ vertices can be covered by 3 connected matchings.
- $n$ vertices can be covered by 17 monochromatic cycles.
- Pokrovskiy '12: $n$ vertices can be covered by 3 monochromatic paths.
Cover problems

t = 3:

- Gyárfás, Ruszinkó, G.S., Szemerédi ’11: \( \geq (1 - \epsilon)n \) vertices can be covered by 3 monochromatic cycles.
- \( n \) vertices can be covered by 3 connected matchings.
- \( n \) vertices can be covered by 17 monochromatic cycles.
- Pokrovskiy ’12: \( n \) vertices can be covered by 3 monochromatic paths.
- Pokrovskiy ’12: The conjecture is not true for any \( t \geq 3 \).

However, in the counterexample all but one vertex can be covered by \( t \) vertex disjoint monochromatic cycles. So perhaps the following weaker conjecture is true:

**Conjecture**

Let \( G \) be a \( t \)-colored graph. Then there exist a constant \( c = c(t) \) and \( t \) vertex disjoint monochromatic cycles of \( G \) that cover at least \( n - c \) vertices.
1st generalization: non-complete graphs, we \( t \)-color a graph \( G \) with \( \alpha(G) = \alpha \). We may define \( f(n, \alpha, s, t, \mathcal{F}) \) in a similar way.

**Conjecture (G.S. ’11)**

\[
f(n, \alpha, t\alpha, t, \mathcal{C}) = n.
\]
1st generalization: non-complete graphs, we $t$-color a graph $G$ with $\alpha(G) = \alpha$. We may define $f(n, \alpha, s, t, \mathcal{F})$ in a similar way.

**Conjecture (G.S. ’11)**

$$f(n, \alpha, t\alpha, t, \mathcal{C}) = n.$$  

For $t = 1$, this is a well-known result of Pósa (and clearly best possible).
Generalized cover problems

1st generalization: non-complete graphs, we $t$-color a graph $G$ with $\alpha(G) = \alpha$. We may define $f(n, \alpha, s, t, \mathcal{F})$ in a similar way.

Conjecture (G.S. ’11)

$$f(n, \alpha, t\alpha, t, C) = n.$$ 

For $t = 1$, this is a well-known result of Pósa (and clearly best possible). For $t = 2$ it would also be best possible. However, we only have an asymptotic result:

Theorem (Balog, Barát, Gerbner, Gyárfás, G.S. ’12)

For every positive $\eta$ and $\alpha$, there exists an $n_0(\eta, \alpha)$ such that the following holds. If $G$ is a 2-colored graph on $n$ vertices, $n \geq n_0$, $\alpha(G) = \alpha$, then there are at most $2\alpha$ vertex disjoint monochromatic cycles covering at least $(1 - \eta)n$ vertices of $V(G)$. 
Generalized cover problems

For a general $t$ we have the following result:

**Theorem (G.S. ’11)**

The vertex set of any $t$-colored $G$ with $\alpha(G) = \alpha$ can be partitioned into at most $25(\alpha t)^2 \log(\alpha t)$ vertex disjoint monochromatic cycles.

Proof idea: Absorbing Method $+$ induction on $\alpha$.

Unfortunately, Pokrovskiy’s counterexample disproves this conjecture as well. Perhaps the following weaker conjecture is true:

**Conjecture**

Let $G$ be a $t$-colored graph with $\alpha(G) = \alpha$. Then there exists a constant $c = c(\alpha, t)$ and $\alpha t$ vertex disjoint monochromatic cycles of $G$ that cover at least $n - c$ vertices.

Pokrovskiy’s counterexample implies that $c \geq \alpha$. 
For a general $t$ we have the following result:

**Theorem (G.S. ’11)**

The vertex set of any $t$-colored $G$ with $\alpha(G) = \alpha$ can be partitioned into at most $25(\alpha t)^2 \log(\alpha t)$ vertex disjoint monochromatic cycles.

Proof idea: Absorbing Method + induction on $\alpha$.

Unfortunately, Pokrovskiy’s counterexample disproves this conjecture as well. Perhaps the following weaker conjecture is true:

**Conjecture**

Let $G$ be a $t$-colored graph with $\alpha(G) = \alpha$. Then there exist a constant $c = c(\alpha, t)$ and $t\alpha$ vertex disjoint monochromatic cycles of $G$ that cover at least $n - c$ vertices.

Pokrovskiy’s counterexample implies that $c \geq \alpha$. 
2nd generalization: non-complete graphs, we $t$-color a graph $G$ with $\delta(G) > \delta$. We may define $f(n, \delta, s, t, F)$ in a similar way.

**Conjecture**

$$f(n, \frac{3n}{4}, 2, 2, C) = n,$$

where again we can partition into 2 monochromatic cycles of different color.

Thus the Bessy-Thomassé result would hold for graphs with minimum degree larger than $3n/4$ (sharp). Again, we only have an asymptotic result:
Generalized cover problems

2nd generalization: non-complete graphs, we $t$-color a graph $G$ with $\delta(G) > \delta$. We may define $f(n, \delta, s, t, \mathcal{F})$ in a similar way.

**Conjecture**

$$f(n, \frac{3n}{4}, 2, 2, C) = n,$$

where again we can partition into 2 monochromatic cycles of different color.

Thus the Bessy-Thomassé result would hold for graphs with minimum degree larger than $3n/4$ (sharp). Again, we only have an asymptotic result:

**Theorem (Balog, Barát, Gerbner, Gyárfás, G.S. ’12)**

For every $\eta > 0$, there is an $n_0(\eta)$ such that if $G$ is a graph on $n \geq n_0$ vertices, $\delta(G) > (\frac{3}{4} + \eta)n$, then every 2-edge-coloring of $G$ admits two vertex disjoint monochromatic cycles of different colors covering at least $(1 - \eta)n$ vertices of $G$. 
Generalized cover problems

3rd generalization: hypergraphs, we $t$-color the edges of the complete $k$-uniform hypergraph $K_n^{(k)}$. We may define $f_k(n, s, t, \mathcal{F})$ in a similar way. Let us consider loose cycles first. The definition is similar for $K_n^{(k)}$.

**Definition**

$C_m$ is a **loose cycle** in $K_n^{(3)}$, if it has vertices $\{v_1, \ldots, v_m\}$ and edges

\[ \{(v_1, v_2, v_3), (v_3, v_4, v_5), (v_5, v_6, v_7), \ldots, (v_{m-1}, v_m, v_1)\} \]

(so in particular $m$ is even).
We have the following result for loose cycles (improving an earlier result):

**Theorem (G.S. ’12)**

For all integers $t, k \geq 2$ there exists a constant $n_0 = n_0(t, k)$ such that if $n \geq n_0$ and the edges of the complete $k$-uniform hypergraph $K_n^{(k)}$ are colored with $t$ colors then the vertex set can be partitioned into at most $50tk \log (tk)$ vertex disjoint monochromatic loose cycles.

The proof is using the Strong Hypergraph Regularity Lemma and the recent Hypergraph Blow-up Lemma of Keevash. We do not risk an exact conjecture here. It would be nice to prove a similar result for tight cycles.
Returning to the original $f(n, s, t, P)$. Many open problems. Let us mention one interesting problem here:

**Conjecture**

$$f(n, 2, 3, P) \sim f(n, 2, 3, C) \sim \frac{6n}{7}.$$  

The reason why we believe this is the following theorem:

**Theorem (Gyárfás, G.S., Selkow '11)**

$$f(n, t - 1, t, M) \sim \frac{(2^t - 2)n}{2^t - 1}, \text{ so } f(n, 2, 3, M) \sim \frac{6n}{7}.$$  

If we could generalize this for $CM$, then we would get the conjecture.
Most of the problems and results mentioned can be found in:


This paper and all of my papers can be downloaded from my homepage: http://web.cs.wpi.edu/~gsarkozy/
Most of the problems and results mentioned can be found in:


This paper and all of my papers can be downloaded from my homepage: http://web.cs.wpi.edu/~gsarkozy/

Thank you!