Monochromatic covers in local edge colorings

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Abstract

An edge coloring of a graph is a local $r$-coloring if the edges incident to any vertex are colored with at most $r$ distinct colors. In this paper, generalizing our earlier work, we study the following problem. Given a family of graphs $\mathcal{F}$ (for example matchings, paths, cycles, powers of cycles and paths, connected subgraphs) and fixed positive integers $s, r$, at least how many vertices can be covered by the vertices of no more than $s$ monochromatic members of $\mathcal{F}$ in every local $r$-coloring of $K_n$. Several problems and results are presented. In particular, we prove the following two results. First, if $n$ is sufficiently large then in any local $r$-coloring of the edges of $K_n$, the vertex set can be partitioned by the vertices of at most $r$ monochromatic trees, which is sharp for local $r$-colorings (unlike for ordinary $r$-colorings according to the Ryser conjecture). Second, we show that we can partition the vertex set with at most $O(r \log r)$ monochromatic cycles in every local $r$-coloring of $K_n$. This answers a question of Conlon and Stein and slightly generalizes one of my favorite joint results with Endre (and with Gyárfás and Ruszinkó).

Dedicated to Endre Szemerédi on the occasion of his 80th birthday

1 Introduction

It is a great honor for me to contribute to this special volume. I have known Endre pretty much all my life. He was my PhD supervisor and he helped a tremendous amount in my career and in my life. Happy birthday Endre! I am grateful for everything!

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1.1 Monochromatic covers in local edge colorings

An edge coloring of a graph is a local \( r \)-coloring if the edges incident to any vertex are colored with at most \( r \) distinct colors. Note that the total number of colors can be much larger than \( r \).

In this paper, generalizing our earlier work, we study the following problem. Given a family of graphs \( \mathcal{F} \) (for example matchings, paths, cycles, powers of cycles and paths, connected subgraphs) and fixed positive integers \( s, r \), at least how many vertices can be covered by the vertices of no more than \( s \) vertex-disjoint monochromatic members of \( \mathcal{F} \) in every local \( r \)-coloring of \( K_n \). More precisely, we are looking for the minimum (for all colorings) of the maximum size of all such covers.

For ordinary \( r \)-colorings (from now on we will just call these \( r \)-colorings as opposed to local \( r \)-colorings) we investigated this problem in [24]. This general problem unites two classical problems, both with a huge literature (see e.g. [24] or [38]). At one end of the spectrum \( (s = 1) \), we have the Ramsey problem, while at the other end we have monochromatic partition problems where we want to cover all the vertices. Mostly we will focus on these two extremal cases. Several of our joint results with Endre fall into these two categories (e.g. [18, 19, 20, 21, 22, 25, 26, 27, 28, 29]).

The generalizations of these two classical problems to local \( r \)-colorings have been studied already; for Ramsey problems e.g. in [5, 9, 16, 17, 39, 40, 41, 42] and for partition problems e.g. in [10, 32].

Next we look at two specific families \( \mathcal{F} \), connected subgraphs and cycles, and study what happens to these two problems in local \( r \)-colorings. Some answers stay the same but most change.

1.1.1 Connected subgraphs

Since every connected component contains a spanning tree of the component, we use here the somewhat simpler tree language.

We start with the Ramsey problem and with \( r \)-colorings. An easy exercise - in fact a note of Paul Erdős - is that in every 2-coloring of the edges of \( K_n \) there is a monochromatic connected subgraph on \( n \) vertices. For three colors the analogue problem was solved in [2, 14]. The problem was rediscovered in [6]. The generalization of this for \( r \) colors is proved by Gyárfás [15]: if the edges of \( K_n \) are colored with \( r \) colors then there is a monochromatic connected component with at least \( \frac{n}{r-1} \) vertices. This result also follows from a more general result of Füredi [12]. The result is sharp if \( r - 1 \) is a prime power and \( r - 1 \) divides \( n \). For sharp results when \( r - 1 \) does not divide \( n \) and \( r \) is small, see [7]. Generalization of the problem for hypergraphs is treated in [13]. There are some results for the case when connectivity is replaced by \( k \)-connectivity [8, 33].
In [23] we showed how the answer changes (and gets slightly smaller than \( \frac{n}{r-1} \)) if \( r \)-coloring is replaced by a local \( r \)-coloring.

**Theorem 1.** ([23]) In any local \( r \)-coloring of a complete graph \( K_n \) there is a monochromatic subtree of size at least \( \frac{nr}{r^2-r+1} \) with equality if a finite projective plane of order \( r-1 \) exists and \( r^2-r+1 \) divides \( n \).

To show equality in the claimed case, consider the points of a finite projective plane of order \( r-1 \) as the vertices of a complete graph and color each pair of vertices by the line going through them. Then replace each vertex \( i \) by a \( k \)-element set \( A_i \) so that the \( A_i \)'s are pairwise disjoint. The coloring is extended naturally with the proviso that the edges within \( A_i \)'s are colored with some color among the colors that were incident to vertex \( i \). The result is a locally \( r \)-colored \( K_n \) where \( n = k(r^2-r+1) \) and the largest monochromatic connected subgraph has \( kr = \frac{nr}{r^2-r+1} \) vertices. For the sake of completeness we present the short proof of Theorem 1 in Section 2.

Let us turn to the other extremal case, monochromatic covers and partitions (in a cover the monochromatic connected subgraphs do not have to be vertex disjoint). A special case of a conjecture attributed to Ryser, (appearing in his student, Henderson’s thesis [31]) states that every intersecting \( r \)-partite hypergraph has a transversal of at most \( r-1 \) elements. Using the dual of the hypergraph of monochromatic components in an \( r \)-coloring of a complete graph, one can easily see that the following form of the conjecture (introduced in [15]) is equivalent.

**Conjecture 1.** (Ryser) In any \( r \)-coloring of the edges of a complete graph, the vertex set can be covered by the vertices of at most \( r-1 \) monochromatic trees.

This was extended further in [11] by changing cover to partition. This is still open. Haxell and Kohayakawa came close to proving it in [30], they showed that \( r \) monochromatic trees suffice if \( n \) is sufficiently large. The proof was simplified and improved in [3].

**Theorem 2.** ([3]) Let \( r \geq 1 \) and \( n \geq 3r^2r!\log r \) be integers. Then in any \( r \)-coloring of the edges of \( K_n \), the vertex set can be partitioned by the vertices of at most \( r \) monochromatic trees of radius at most 2, each of different color.

Perhaps somewhat surprisingly for local \( r \)-colorings this is the precise answer (again for sufficiently large \( n \)).

**Theorem 3.** Let \( r \geq 1 \) and \( n \geq 3r^2r!\log r \) be integers. Then in any local \( r \)-coloring of the edges of \( K_n \), the vertex set can be partitioned by the vertices of at most \( r \) monochromatic trees of radius at most 2, each of different color.
In light of Theorem 1 this is clearly sharp; \( r - 1 \) trees might not suffice. The proof is using the proof method of [3] adapted to local \( r \)-colorings.

It is not clear what happens “in-between”, i.e. how much can we cover with \( s \) vertex-disjoint monochromatic trees in any local \( r \)-coloring, where \( 1 < s < r \).

1.1.2 Cycles

The Ramsey problem for cycles has a huge literature (see [38]). For local \( r \)-colorings the only result we found is the following result of Conlon and Stein [10] (we shall need this later).

**Theorem 4.** (Corollary 2.5 in [10]) In any local \( r \)-coloring of the edges of \( K_n \) there is a monochromatic cycle of length at least \( n/2r \).

It is not clear how far this is from the optimal.

Let us turn to partitions. The main open problem of the area is the famous conjecture of Erdős, Gyárfás and Pyber [11] claiming that the vertex set of every \( r \)-colored complete graph can be partitioned into \( r \) monochromatic cycles, where the empty set, an edge and a vertex are considered degenerate cycles. This conjecture was proved for \( r = 2 \) (this special case was conjectured earlier by Lehel) and \( n \geq n_0 \) by Luczak, Rödl and Szemerédi [34]. First the value of \( n_0 \) was improved by Allen [1], then Bessy and Thomassé [4] found an elementary argument that works for every \( n \). For \( r = 3 \) we proved the conjecture asymptotically in [22]. However, Pokrovskiy [36] found a counterexample to the conjecture for any \( r \geq 3 \). The counterexample is quite weak in the sense that all but one vertex can be covered by \( r \) vertex disjoint monochromatic cycles. Thus perhaps a slightly weaker version of the conjecture still can be true, that apart from a constant number of vertices all vertices can be covered by \( r \) vertex disjoint monochromatic cycles. The generalization of this problem to local \( r \)-colorings was initiated by Conlon and Stein [10] who proved that the above Bessy-Thomassé result can be generalized to local 2-colorings. For a general \( r \) in [10] they showed that the vertex set of every locally \( r \)-colored complete graph can be partitioned into at most \( O(r^2 \log r) \) monochromatic cycles. They asked for an improvement of this; in particular whether the same \( O(r \log r) \) bound from Theorem 5 can be proved for local \( r \)-colorings. Lang
and Stein [32] improved the bound to $O(r^2)$. Here we show that indeed the $O(r \log r)$ bound holds for local $r$-colorings as well.

**Theorem 6.** For every integer $r \geq 2$ there exists a constant $n_0 = n_0(r)$ such that if $n \geq n_0$ then the vertex set of every locally $r$-colored complete graph can be partitioned into at most $200r \log r$ monochromatic cycles.

The proof is using the proof method of [18] adapted to local $r$-colorings.

Probably this is true with $O(r)$, but this is already open for ordinary $r$-colorings. Again it is not clear what happens “in-between”, i.e. how much can we cover with $s$ vertex-disjoint monochromatic cycles in any local $r$-coloring, where $1 < s < r$. Furthermore, it would be interesting to study some other families, for example matchings, where there are some results available for $r$-colorings (see [24]).

In Section 2 we give the proofs for connected subgraphs and in Section 3 for cycles.

## 2 Proofs for connected subgraphs

We start with the simple proof of Theorem 1 from [23]. It is based on the following result for bipartite graphs. A **double star** is a tree obtained from two vertex disjoint stars by connecting their centers.

**Theorem 7.** ([23]) Assume that the edges of a complete bipartite graph $G = [A, B]$ are colored so that the edges incident to any vertex of $A$ are colored with at most $p$ colors and the edges incident to any vertex of $B$ are colored with at most $q$ colors. Then there exists a monochromatic connected subgraph $H$ with at least $|A|/q + |B|/p$ vertices. In fact, $H$ can be selected as a double star.

**Corollary 1.** [23] If the edges of a complete bipartite graph $G$ are locally $r$-colored, there exists a monochromatic connected subgraph (in fact a double star) with at least $|V(G)|/r$ vertices.

The special case of Corollary 1, when local $r$-colorings are replaced by $r$-colorings was proved in [15] (without the remark about the double star). A considerably simpler proof (that gives the stronger result about the double star) was given by Liu, Morris and Prince [33] (this was also proved independently by Mubayi [35]). We use the method in [33] to prove Theorem 7.

**Proof of Theorem 7:** Let $N_i(v)$ denote the neighborhood of $v$ in color $i$ and let $d_i(v) = |N_i(v)|$. For any edge $ab$ of color $i$, $a \in A, b \in B$, set $c(a, b) = d_i(a) + d_i(b)$. Let $I(v)$ denote the set of colors on the edges incident to $v \in V(G)$. Then, by using the Cauchy-Swartz inequality and the local coloring conditions, we get

$$
\sum_{ab \in E(G)} c(a, b) = \sum_{a \in A} \sum_{i \in I(a)} d_i^2(a) + \sum_{b \in B} \sum_{i \in I(b)} d_i^2(b) \geq |A|p \left( \sum_{a \in A} \sum_{i \in I(a)} d_i(a) \right)^2 + \left( \sum_{b \in B} \sum_{i \in I(b)} d_i(b) \right)^2.
$$
at least then stop as we would already have the desired tree partition. So some vertex \( x \) of \( N \) has at most \( Z = \sum_{i \in I(b)} d_i(b) \) partition \( \{ A, B \} \). Therefore for some \( a \) if for all \( v \in A \) are colored with at most \( p \) colors and the edges incident to any \( v \in B \) are colored with at most \( q = r \) colors. Thus, by Theorem 5, there is a monochromatic component of size at least

\[
|A|/q + |B|/p = |A| + n - |A| = n - |A| \left( \frac{1}{r - 1} - \frac{1}{r} \right) \geq \frac{n}{r - 1} - \frac{r n}{r - 1} \frac{1}{r^2 - r + 1}.
\]

□

Proof of Theorem 1: If any monochromatic, say red component \( C \) satisfies \( |C| \geq \frac{rn}{r^2 - r + 1} \), we have nothing to prove. Otherwise apply Theorem 5 for the complete bipartite graph \([ A, B \] = [V(C), V(G) \setminus V(C)]\). The edges incident to any \( v \in A \) are colored with at most \( p = r - 1 \) colors and the edges incident to any \( v \in B \) are colored with at most \( q = r \) colors. Thus, by Theorem 5, there is a monochromatic component of size at least

\[
|A|/q + |B|/p = |A| + n - |A| = n - |A| \left( \frac{1}{r - 1} - \frac{1}{r} \right) \geq \frac{n}{r - 1} - \frac{r n}{r - 1} \frac{1}{r^2 - r + 1}.
\]

□

Proof of Theorem 3: We may assume \( r \geq 2 \), otherwise the statement is trivial. We tailor the proof of Bal and DeBiasio [3] to our needs.

We need the following lemma from [3].

Lemma 1. ([3]) Let \( k \geq 2 \). If \( G \) is an \([ Y, Z ]\)-bipartite graph such that for all \( v \in Z \), \( \deg(v, Y) \geq k \log |Z| \), then in every \( k \)-coloring of the edges of \( G \), there exists a partition \( \{ Y_1, \ldots, Y_k \} \) of \( Y \) such that for all \( v \in Z \), there exists \( i \in [k] \) such that \( N_i(v) \cap Y_i \neq \emptyset \).

The proof of Theorem 3 is in two steps.

Step 1: Let \( x_1 \in V(G) \) and let \( Y_1 \) be the largest monochromatic neighborhood of \( x_1 \), say the color is 1. Note that \( |Y_1| \geq (n - 1)/r \) (since we have a local \( r \)-coloring, \( x_1 \) sees at most \( r \) colors). If every vertex in \( V \setminus Y_1 \) has a neighbor of color 1 in \( Y_1 \), then stop as we would already have the desired tree partition. So some vertex \( x_2 \) has at least \( \frac{1}{r - 1} |Y_1| \) neighbors of say color 2 in \( Y_1 \) (since again \( x_2 \) sees at most \( r \) colors). Set \( Y_2 := Y_1 \cap N_2(x_2) \).

For \( 2 \leq i \leq r - 1 \), assuming \( Y_i \) has already been dened, we do the following: if for all \( v \in V \setminus Y_i \), \(|(\cup_{j=1}^{i} N_j(v)) \cap Y_i| > i \log n \), then set \( k := i \), \( Y := Y_k \), and \( Z = V \setminus (\{x_1, \ldots, x_k\} \cup Y_k) \) then proceed to Step 2. Otherwise some vertex \( x_{i+1} \in V \setminus Y_i \) has at most \( i \log n \) neighbors having colors from \([i]\) in \( Y_i \) and thus \( x_{i+1} \) has at least \( \frac{1}{r - 1} (|Y_i| - i \log n) \) neighbors of color say \( i + 1 \), in \( Y_i \) (since \( x_{i+1} \) sees at most \( r \) colors).
Set $Y_{i+1} := Y_i \cap N_{i+1}(x_{i+1})$. Continue in this manner until we go to Step 2 or until $Y_r$ has been defined. After we complete the $i = (r - 1)$-th step, we have

$$|Y_r| \geq \frac{n - 1}{r!} - \log n \sum_{j=1}^{r-2} \frac{r - j}{j!} \geq r \log n,$$

(for details of the computation why this is true, see [3]). Now set $k := r$, $Y := Y_r$, and $Z = V \setminus \{x_1, \ldots, x_r \cup Y_r\}$ then proceed to Step 2.

**Step 2:** Note that for all $v \in Z$, $|\bigcup_{j=1}^k N_j(v) \cap Y_k| \geq k \log n > k \log |Z|$ (if $k = r$, this is true since each vertex in $Y$ sees all of the colors in $[r]$ and thus every edge in the bipartite graph $[Y, Z]$ must have a color in $[r]$). Thus we may apply Lemma 1 to get a partition $\{Y'_1, \ldots, Y'_k\}$ of $Y$ such that for all $v \in Z$, there exists $i \in [k]$ such that $N_i(v) \cap Y'_i \neq \emptyset$. For each $v \in Z$ we choose an arbitrary such $i$ and an arbitrary neighbor in $N_i(v) \cap Y'_i$. Then $x_i$ along with $Y'_i$ and all the $v \in Z$ which chose neighbors in $Y'_i$ form a tree of color $i$ and radius at most 2. Thus we have a partition into $k \leq r$ monochromatic trees.

\[ \square \]

3 **Proofs for cycles**

**Proof of Theorem 6:** We follow closely the proof method in [18] and in [19] (part of the proof was improved in [19]) and we adapt it to local $r$-colorings. Since the proof is fairly complicated, we are not going to give the whole proof again, we just focus on the modifications we have to make for local $r$-colorings. All the other details are unchanged and can be found in [18] and in [19].

First we have to “localize” the Regularity method, i.e. we have to adapt it to local $r$-colorings; this is done similarly as in [32]. While the total number of colors may be very large compared to $r$, we may restrict our attention to the $r'$ colors with the largest density, where $r'$ depends only on $r$ and $\varepsilon$. Indeed, we will use the following two lemmas.

**Lemma 2.** ([17]) Let a graph $G$ with average degree $d$ be locally $r$-colored. Then one color has at least $d^2/2r^2$ edges.

Then using this the following is proved in [32].

**Lemma 3.** ([32]) For all $\varepsilon > 0$ and integer $r \geq 1$ there is a $r' = r'(\varepsilon, r)$ such that for any local $r$-coloring of $K_n$, there are $r'$ colors such that all but at most $\varepsilon n^2$ edges use these colors.

Then we may use the $r'$-color version of the Regularity Lemma (Lemma 1 in [18]). The proof in [18] is in 4 steps; we go through the steps and show what modifications we have to make in each step for local $r$-colorings.
3.1 Step 1

Consider a local $r$-edge coloring $(G_1, G_2, \ldots, G_s)$ of $K_n$, where $s$ is the total number of colors. Applying Lemma 3, there is a $r' = r'(\varepsilon, r)$ such that the union of $G_1, \ldots, G_{r'}$ covers all but at most $\varepsilon n^2/8r^2$ edges of $K_n$. We merge the remaining edges into a new color $G_0 := \cup_{i=r'+1}^s G_i$. Note that this is still a local $r$-coloring and by the choice of $r'$, we have

$$|E(G_0)| \leq \varepsilon n^2/8r^2. \quad (1)$$

Apply the $(r' + 1)$-color version of the Regularity Lemma (Lemma 1 in [18]), with $\varepsilon \ll 1$, and $m = 1/\varepsilon$ to get a partition of $V(K_n) = V = \cup_{0 \leq i \leq l} V_i$, where $|V_i| = m, 1 \leq i \leq l$. We define the reduced graph $G^R$: The vertices of $G^R$ are $p_1, \ldots, p_l$, and we have an edge between vertices $p_i$ and $p_j$ if the pair $\{V_i, V_j\}$ is $(\varepsilon, G_q)$-regular for $q = 0, 1, \ldots, r'$. Thus we have a one-to-one correspondence $f : p_i \to V_i$ between the vertices of $G^R$ and the clusters of the partition. Then,

$$|E(G^R)| \geq (1 - \varepsilon) \left(\frac{l}{2}\right),$$

and thus $G^R$ is a $(1 - \varepsilon)$-dense graph on $l$ vertices.

Define an edge-coloring $(G^R_0, G^R_1, \ldots, G^R_{r'})$ of $G^R$ by the $r'+1$ colors in the following way. The edge $p_i p_j$ is colored with a color $q$ that contains the most edges from $K(V_i, V_j)$, thus from Lemma 2 we have

$$|E_{G_q}(V_i, V_j)| \geq \frac{1}{2r^2} \left(\frac{n}{2l}\right)^2. \quad (2)$$

Then we have the following.

**Claim 1.** (*Claim 2.9 in [32]*) This coloring of $G^R$ is a local $r$-coloring.

Furthermore, by (1) and (2), we know that $G^R$ has at most $|E(G_0)| \frac{4\varepsilon^2 r^2}{m^2} \leq \varepsilon l^2$ edges in color $G_0$. Delete these edges from $G^R$, so now we have an $r'$-coloring which is a local $r$-coloring. Let us take the color class in this coloring that has the most edges. For simplicity assume that this is $G^R_1$ and call this color red. Using Lemma 2 the average degree in $G^R_1$ is at least $l/4r^2$. Then using Lemma 4 in [18] we can find a connected $l/32r^2$-half dense matching $M$ in $G^R_1$ (instead of the $l/16r$ in [18]). The rest of Step 1 is the same except from (2) the density $1/r$ in a regular pair is replaced with $1/2r^2$ everywhere.

3.2 Step 2

Here the main difference is that instead of the Erdős-Gallai theorem we use Theorem 4. We go back from the reduced graph to the original graph and we remove the
vertices assigned to the matching $M$, i.e. $f(M)$. We apply repeatedly Theorem 4 to the locally $r$-colored complete graph induced by $K_n \setminus f(M)$. We also change the constant $c$ to $c = 1/150r^2$ (from $c = 1/350r$). (Note that still $c < 0.001$ using $r \geq 3$, as required in Step 3.)

Then as in [18] we get the inequality
\[
(n - |f(M)|) \left(1 - \frac{1}{2r}\right)^t \leq c^{12}n.
\]
This inequality is certainly true if
\[
\left(1 - \frac{1}{2r}\right)^t \leq c^{12},
\]
which in turn is true using $1 - x \leq e^{-x}$ if
\[
e^{-\frac{t}{2r}} \leq c^{12}.
\]
This shows that we can choose $t = 24r \lceil \log (150r^2) \rceil$.

### 3.3 Steps 3 and 4

These steps are exactly the same as in [18] except we have to use the following bipartite lemma instead of Lemma 6 in [18]. This is the localized version of the corresponding lemma in the improved proof [19] (Theorem 1.2 in [19]).

**Lemma 4.** For every fixed $r$ there exists $n_0 = n_0(r)$ such that the following is true. Assume that we have a local $r$-coloring of the complete bipartite graph $K(A, B)$, where $|A| \geq n_0$. If $|A| \geq 2r|B|$, then $B$ can be covered by at most $3r$ vertex disjoint monochromatic cycles.

We prove this lemma in the next section. Thus similarly as in [18], the total number of vertex disjoint monochromatic cycles we used to partition the vertex set of $K_n$ is at most
\[
24r \lceil \log (150r^2) \rceil + 3r + 3r + 2 \leq 200r \log r,
\]
finishing the proof of Theorem 1. □
3.4 Proof of Lemma 4

Here we prove Lemma 4. First we need the localized version of the main lemma in [19] (Theorem 2.1 in [19]). Note that while the original version of this lemma appeared in the joint paper [19], it would be hard to hide the fact that it is all vintage Endre. I still remember when one day he came in to SZTAKI and explained this beautiful argument to us.

Lemma 5. Assume that the edges of a complete bipartite graph $K(A, B)$ are locally $r$-colored and $|A| \geq r|B|$. Then there are vertex disjoint monochromatic connected matchings, all of different color, such that their union covers each vertex of $B$.

Proof: Since the proof is so short and elegant we give the whole localized proof not just the modifications. We define by iteration $r$-colored complete bipartite graphs $G_i = K(A, B)$ and sets $X_i \subseteq A, Y_i \subseteq B$, such that $A_i = \cup_{j=0}^i X_j$. Initially $G_0 = G, A_0 = X_0 = Y_0 = \emptyset$.

The general step is to select an arbitrary vertex $a \in A \setminus A_{i-1}$ and consider the partition $P$ of $B$ by putting two vertices $p, q \in B$ into the same class if and only if the colors of $ap, aq$ are the same and label the class by the color of $ap$. Note that there are at most $r$ classes since the coloring is a local $r$-coloring, so $a$ is incident to edges only in at most $r$ colors. Let $E$ be defined as the set of those edges $ab$ of $G_{i-1}$ whose color is the same as the label of the class of $P$ containing $b$. Observe that the existence of a matching of $B$ to $A \setminus A_{i-1}$ using edges of $E$ proves the theorem - then the procedure stops. Therefore we may assume that such a matching does not exist. By Hall’s theorem there are sets $X_i \subseteq (A \setminus A_{i-1}), Y_i \subseteq B$ such that $|X_i| < |Y_i|$ and all edges of $E$ incident to $Y_i$ are incident to $X_i$ (i.e. $X_i$ is the set of $E$-neighbors of $Y_i$). Set $A_i = A_{i-1} \cup X_i$ and let $G_i$ be the complete bipartite subgraph of $G$ spanned by $[A \setminus A_i, B]$. Notice that $a \in X_i$ thus at least one new vertex is added to $A_i$. This finishes the definitions for step $i$.

Since at each step $|A_i| > |A_{i-1}|$, the procedure terminates with $A_m = A$ (and $G_m = \emptyset$) for some $m$. We show that this leads to a contradiction, thus the procedure must terminate with finding the required cover of $B$.

Assume that a vertex $b \in B$ in $G_m$ is covered by $k$ of the sets $Y_i$, w.l.o.g by $Y_1, Y_2, \ldots, Y_k$. Then there are $k$ distinct colors such that all edges incident to $b$ in one of these colors go to $\cup_{i=1}^k X_i$. Therefore $b$ is incident to edges of at most $r - k$ colors in $G_k$ (since again $b$ is incident to edges in only at most $r$ colors) implying $k \leq r$. Assuming that the procedure takes $m$ steps, consider the hypergraph on vertex set $B$ with edges $Y_i$,

$$r|B| \geq \sum_{x \in B} d(x) = \sum_{j=1}^m |Y_j| \geq \sum_{j=1}^m (|X_j| + 1) = |A| + m > |A| \quad (3)$$
contradicting the assumption of the theorem. □

Then we can use the localized Regularity method, as described above, the same way as in [19] to turn the connected matchings into cycles. This gives us a covering of $B$ with at most $r$ vertex disjoint monochromatic cycles that cover $B$ apart from at most $2\sqrt{\varepsilon|B|}$ vertices. For the covering of these remaining vertices we can apply the following lemma (the localized version of Lemma 8 in [18]).

**Lemma 6.** There exists a constant $n_0$ such that the following is true. Assume that the edges of the complete bipartite graph $K(A, B)$ are locally $r$-colored. If $|A| \geq n_0$, $|B| \leq |A|/(8r)^{10(r+1)}$, then $B$ can be covered by at most $2r$ vertex disjoint monochromatic cycles.

Indeed we can apply this lemma as $\varepsilon$ is sufficiently small. Thus altogether we covered $B$ with at most $r + 2r = 3r$ vertex disjoint monochromatic cycles, and thus finishing the proof of Lemma 4.

As in [18] in the proof of Lemma 6 we will also use the following simple lemma about the case when $B$ is significantly smaller than $A$ (the localized version of Lemma 9 in [18]).

**Lemma 7.** Assume that the edges of the complete bipartite graph $K(A, B)$ are colored with $r$ colors. If $(|B| - 1)r^{|B|} < |A|$, then $B$ can be covered by at most $r$ vertex disjoint monochromatic cycles.

Let us start with the simple proof of this last lemma.

**Proof of Lemma 7:** Denote the vertices of $B$ by $\{b_1, b_2, \ldots, b_{|B|}\}$. To each vertex $v \in A$ we assign a vector $(v_1, v_2, \ldots, v_{|B|})$ of colors, where $v_i$ is the color of the edge $(v, b_i)$. Note that since we have a local $r$-coloring, the number of distinct colors in $(v_1, v_2, \ldots, v_{|B|})$ is at most $r$. Furthermore, again because we have a local $r$-coloring, the total number of distinct color vectors possible is $r^{|B|}$, since $v_i$ can be one of at most $r$ colors. Since we have $|A| > (|B| - 1)r^{|B|}$ vectors, by the pigeon-hole principle we must have a vector that is repeated at least

$$\frac{|A|}{r^{|B|}} \geq |B|$$

times. In other words, there are at least $|B|$ vertices in $A$ for which the colorings of the edges going to $\{b_1, b_2, \ldots, b_{|B|}\}$ are exactly the same and by the above this coloring is using at most $r$ colors. It is easy to see that this gives a covering of $B$ by at most $r$ vertex disjoint monochromatic cycles (in fact complete bipartite graphs can be used instead cycles). □

**Proof of Lemma 6:** We proceed exactly as in the proof of Lemma 8 in [18] but using the localized version of the Regularity method as described in Step 1 above. As
in Step 1, there are some minor, numerical adjustments we have to make due to the fact that the density of the most frequent color class is $1/2r^2$ instead of $1/r$. Here is the modified version of Claim 1 in the proof of Lemma 8 in [18].

**Claim 2.** There exists a color (say $G_1$, called red) such that $G^R_1 = (A^R, B^R)$ contains a connected $(A''^R, B''^R)$ satisfying the following:

$$|A''^R| \geq \frac{1}{8^2(2r)^8}l_A \text{ and } |B''^R| \geq \frac{1}{8(2r)^4}l_B,$$

(4)

$$\deg_{G^R_1}(p^j_B, A^R) \geq \frac{1}{2(2r)^2}l_A, \ \forall p^j_B \in B''^R,$$

(5)

$$\deg_{G^R_1}(p^j_A, B''^R) \geq \frac{1}{8(2r)^4}l_B, \ \forall p^j_A \in A''^R.$$

(6)

Then we have to carry these changes through. (15) in [18] is modified to

$$\deg_{G^R_1}(p^j_B, A''^R) \geq \frac{2}{(8r)^{10(r+1)}}l_A (\geq l_B),$$

(16) is modified to

$$\deg_{G_1}(v, f(A''^R)) \geq \frac{4}{(8r)^{9(r+1)}}|A| (\geq |B|),$$

and then the number of vertices $v$ satisfying this is at most $\frac{2}{(8r)^{10(r+1)}}|B|$ in (17).

The only other change we have to make in the proof is the following. When we remove the red cycle just before Case 1, $B_1$ still denotes the remaining set in $B$ but to define $A_1$ in $f(A''^R)$ we keep only those remaining vertices which are incident to at least one red edge. $\varepsilon$-regularity implies that most vertices are such in $f(A''^R)$. Indeed, consider an arbitrary cluster $p^j_A \in A''^R$. By Claim 2, $p^j_A$ has a red neighbor $p^j_B \in B''^R$ (it actually has many). Then the $\varepsilon$-regularity of the pair $(p^j_A, p^j_B)$ implies that apart from an $\varepsilon$-fraction, all vertices in $f(p^j_A)$ have many red neighbors in $f(p^j_B)$.

(18) in [18] is modified to

$$|A_1| \geq \frac{1}{(8r)^{8}|A_0|}.$$

We proceed with Cases 1,2 and 3 exactly as in [18], but $(8r)^{8(r+1)}$ is replaced with $(8r)^{10(r+1)}$. In the next iteration when we apply the localized Regularity method again between $A_1$ and $B_1$, we remove first the red edges (they have a small density).

We finish almost as in [18]. If the procedure terminates after $k(\leq r)$ iterations with no more vertices remaining in $B$ (i.e. we had a Case 1), then we have a cover of $B$ with at most $2r$ vertex disjoint monochromatic cycles, as desired. Assuming
that the procedure does not terminate after $r$ iterations, so $B_r \neq \emptyset$, we will get a contradiction. Indeed, by the construction every vertex in $A_r$ is incident to edges in the same $r$ colors, say colors $i, 1 \leq i \leq r$. But then all the edges between $A_r$ and $B_r$ must be colored with these $r$ colors, since we have a local $r$-coloring. However, then as in [18] for each vertex $v \in B_r$ and color $1 \leq i \leq r$ we have

$$\deg_{G_i}(v, A_r) < \frac{4(2)^r(8r)^{8r}r}{(8r)^{9(r+1)}} |A_r| < \frac{|A_r|}{r},$$

a contradiction, since in at least one of the colors we must have at least $|A_r|/r$ edges from $v$ to $A_r$. This finishes the proof of Lemma 6. □

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**References**


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