Improved monochromatic loose cycle partitions in
hypergraphs *

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December 6, 2012

Abstract

Improving our earlier result we show that every large enough complete $k$-
uniform $r$-colored hypergraph can be partitioned into at most $50rk\log (rk)$
vertex disjoint monochromatic loose cycles. The proof is using the strong hy-
pergraph Regularity Lemma due to Rödl and Schacht and the new, powerful
hypergraph Blow-up Lemma of Keevash.

1 Monochromatic cycle partitions

Assume first that $K_n$ is a complete graph on $n$ vertices whose edges are colored with
$r$ colors ($r \geq 1$). How many monochromatic cycles are needed to partition the vertex
set of $K_n$? This question received a lot of attention in the last few years. Throughout
the paper, single vertices and edges are considered to be cycles. Let $p(r)$ denote the
minimum number of monochromatic cycles needed to partition the vertex set of any
$r$-colored $K_n$. It is not obvious that $p(r)$ is a well-defined function. That is, it is

*2010 Mathematics Subject Classification: 05C55, 05C38.
The author was supported in part by NSF Grant DMS-0968699
not obvious that there always is a partition whose cardinality is independent of the order of the complete graph. However, in [4] Erdős, Gyárfás and Pyber proved that there exists a constant \( c \) such that \( p(r) \leq cr^4 \log r \) (throughout this paper \( \log \) denotes natural logarithm). Furthermore, in [4] (see also [8]) the authors conjectured the following.

**Conjecture 1.** \( p(r) = r \).

The special case \( r = 2 \) of this conjecture was asked earlier by Lehel and for \( n \geq n_0 \) was first proved by Łuczak, Rödl and Szemerédi [24]. Allen improved on the value of \( n_0 \) [1] and finally recently Bessy and Thomasse [3] proved the original conjecture for \( r = 2 \). For general \( r \) the current best bound is due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [9] who proved that for \( n \geq n_0(r) \) we have \( p(r) \leq 100r \log r \). For \( r = 3 \) the conjecture was asymptotically proved in [11] but surprisingly Pokrovskiy [25] found a counterexample to that conjecture. However, in the counterexample all but one vertex can be covered by \( r \) vertex disjoint monochromatic cycles. Thus a slightly weaker version of the conjecture still can be true, say that apart from a constant number of vertices the vertex set can be covered by \( r \) vertex disjoint monochromatic cycles.

Let us also note that the above problem was generalized in various directions; for complete bipartite graphs (see [4] and [16]), for graphs which are not necessarily complete (see [2] and [29]) and for vertex partitions by monochromatic connected \( k \)-regular subgraphs (see [30] and [31]).

In this paper we study the generalization of this problem for \( r \)-edge colorings of hypergraphs. This question was initiated in [12]. We will consider two kinds of hypergraph cycles: a *loose* cycle in a \( k \)-uniform hypergraph is a sequence of edges, \( e_1, \ldots, e_t \) such that for \( 1 \leq i \leq t \), \( e_i \cap e_{i+1} = v_i \) where \( e_{t+1} = e_1 \) and all the \( v_i \)-s are distinct. A *tight* cycle is a sequence of \( t \) vertices and every consecutive set of \( k \) vertices forms an edge.

In [12] we proved that for all integers \( r \geq 1, k \geq 3 \) there exists a constant \( c = c(r, k) \) such that in every \( r \)-coloring of the edges of the complete \( k \)-uniform hypergraph \( K_n^k \) the vertex set can be partitioned into at most \( c(r, k) \) vertex disjoint monochromatic loose cycles. Thus again we have the same phenomenon as for graphs; the partition number does not depend on the order of the hypergraph. Here we give a significant improvement of this result.

**Theorem 1.** For all integers \( r, k \geq 2 \) there exists a constant \( n_0 = n_0(r, k) \) such that if \( n \geq n_0 \) and the edges of the complete \( k \)-uniform hypergraph \( K_n^k \) are colored with \( r \) colors then the vertex set can be partitioned into at most \( 50rk \log (rk) \) vertex disjoint monochromatic loose cycles.
One of the key tools in the proof is the following $r$-color Ramsey result on loose cycles which may be interesting on its own so we state it here separately (see [13] and [17] for Ramsey results of loose cycles for two colors).

**Lemma 1.** For all $\varepsilon > 0$ and integers $r, k \geq 2$ there exists a constant $n_0 = n_0(r, k, \varepsilon)$ such that if $n \geq n_0$ and the edges of the complete $k$-uniform hypergraph $K_n^{(k)}$ are colored with $r$ colors then there is a monochromatic loose cycle covering at least $(1 - \varepsilon)\frac{n}{rk}$ vertices.

Note that there is an extensive literature on density results for hypergraph paths and cycles: e.g. for paths in [14] and in [26] (see Lemma 4 below), for Berge-cycles in [15], exact results for short loose paths in [5] and for Hamiltonian cycles in [19], [26]. However, as far as we know Lemma 1 is a new result.

We believe that there is room for improvement in both results but we do not risk exact conjectures here. Furthermore, it would be nice to obtain similar results for tight cycles.

**2 Sketch of the proof of Theorem 1**

We may assume throughout that $k \geq 3$, since for graphs we have the result in [9]. A matching in a $k$-uniform hypergraph (or $k$-graph) $H$ is called connected if its edges are all in the same connected component of $H$, where two vertices are in the same connected component if there is a walk between them. The matching is self-connected if the connecting walks can be found within the vertex set of the matching.

To prove Theorem 1 we will follow our proof technique from [9] which is using the Regularity Lemma of Szemerédi [32] and the graph Blow-up Lemma [20, 21]. However, here we have to adapt the method to hypergraphs, so we are going to use the technique from [19] (where they find loose Hamiltonian cycles in dense hypergraphs) which in turn is using the strong hypergraph Regularity Lemma due to Rödl and Schacht [27] and the new, powerful hypergraph Blow-up Lemma of Keevash [18].

Consider an $r$-edge coloring $(H_1, H_2, \ldots, H_r)$ of $K_n^{(k)}$. Let us take the color in this coloring that has the most edges. For simplicity assume that this is $H_1$ and call this color red. We apply the strong hypergraph Regularity Lemma to $H_1$. Then we introduce the so called reduced hypergraph $R$, the hypergraph whose vertices are associated to the clusters in the partition. Then we find a large self-connected matching in this reduced graph. The study of connected matchings for graphs was initiated in [23] and played an important role in several recent papers (e.g. [9], [10]).

Following our method from [9] we establish the bound on the number of monochromatic loose cycles needed to partition the vertex set in the following steps.
• Step 1: We find a sufficiently large self-connected matching \( CM \) in \( R \). It is implicitly proved in [19] using the hypergraph Blow-up Lemma of Keevash that we can remove a small number of exceptional vertices from each cluster of \( CM \) so that there is a spanning loose red cycle in the remainder of \( CM \) and this remains true even if we remove an additional small number of vertices from each cluster of \( CM \) (assuming that the total number of remaining vertices is divisible by \( k - 1 \)).

• Step 2: We remove the non-exceptional vertices of \( CM \) from \( K^{(k)}(n) \) and we go back to the original hypergraph (instead of the reduced hypergraph). We greedily remove a number (depending on \( r \) and \( k \)) of vertex disjoint monochromatic loose cycles from the remainder in \( K^{(k)}(n) \) until the number of leftover vertices is much smaller than the number of vertices associated to \( CM \).

• Step 3: Using a lemma about loose cycle covers of \( r \)-colored unbalanced complete “bipartite” \( k \)-uniform hypergraphs we combine the leftover vertices with some vertices of the clusters associated with vertices of \( CM \).

• Step 4: We remove an additional at most \( k - 2 \) vertices (degenerate loose cycles) from the vertices of \( CM \) to make sure that the total number of remaining vertices is divisible by \( k - 1 \). Finally we find a red cycle spanning the remaining vertices of \( CM \) (using the above remark).

The organization of the proof follows this outline. After giving the definitions and tools, we discuss each step one by one. Since we are using techniques from [9] and [19] at some places we are not going into great details. The missing details can be found in these papers.

3 Tools

3.1 Strong hypergraph regularity and the hypergraph blow-up lemma

In this section we will state the version of the strong hypergraph Regularity Lemma we will use and the hypergraph Blow-up Lemma. We follow the notation, terminology and discussion from [19]. Since this section is quite technical we ask the reader to consult the definitions from [19] that are necessary for the following statements.

First we state the version of the strong hypergraph Regularity Lemma due to Rödl and Schacht we will need. It states that for any large enough \( k \)-graph \( H \) there exists another \( k \)-graph \( G \) that is close to \( H \) and which is regular with respect to some partition complex. There are other versions of the strong hypergraph Regularity Lemma
which give information on $H$ itself (see [28], [6]) but these do not have the density conditions that are needed for the hypergraph Blow-up Lemma (see the discussion in [18]).

**Lemma 2 (Theorem 2.2 in [27], see also Theorem 3.1 in [19]).** Suppose integers $n, t, a, k$ and reals $\epsilon, \nu$ satisfy $0 < 1/n < \epsilon < 1/a < \nu, 1/t, 1/k$ and all divides $n$. Suppose also that $H$ is a $t$-partite $k$-graph whose vertex classes $X_1, \ldots, X_t$ form an equitable partition of its vertex set $X$, where $|X| = n$. Then there is an $a$-bounded $\epsilon$-regular vertex-equitable partition $(k - 1)$-complex $P$ on $X$ and a $t$-partite $k$-graph $G$ on $X$ that is $\nu$-close to $H$ and perfectly $\epsilon$-regular with respect to $P$.

Here $a \ll b$ means that $a$ is sufficiently small compared to $b$.

The other main tool we will need in this section is the recent hypergraph Blow-up Lemma of Keevash. In this result there is not only a $k$-complex $G$, but also a $k$-graph $M$ of “marked” or forbidden edges on the same vertex set. We will find an embedding of any spanning bounded-degree $k$-complex in $G \setminus M$, and thus not using any of the marked edges. We will apply this with $M = G \setminus H$ where $G$ is the $k$-graph given by Lemma 2, and thus we will find the embedding within the original hypergraph $H$. The use of the Blow-up Lemma will be hidden through the following definition.

**Definition 1 (Robustly universal pairs).** Suppose that $G'$ is a $k$-partite $k$-complex on $V' = V'_1 \cup \ldots V'_k$ and $M' \subseteq G'_w$. We say that $(G', M')$ is $(c, c_0, \delta^*)$-robustly $D$-universal if $d(G'_w) > \delta^*$ and $|G'(v)_w| > \delta^*|G'_w|/|V'_j|$ for any $j \in [k], v \in V'_j$, and whenever

(i) $V_j \subseteq V'_j$ are sets with $|V_j| \geq c|V'_j|$ for all $j \in [k]$, such that writing $V = \cup_{j \in [k]} V_j$, $G = G[V], M = M[V]$, we have $|G(v)_w| \geq c|G'(v)_w|$ for any $j \in [k], v \in V'_j$,

(ii) $L$ is a $k$-partite $k$-complex of maximum degree at most $D$ on some vertex set $U = U_1 \cup \ldots U_k$ with $|U_j| = |V_j|$ for all $j \in [k]$,

(iii) $Z_u \subseteq V(u)$ are sets with $|Z_u| \geq c|V(u)|$ for every $u$ in some $U_* \subseteq U$ such that, writing $U'_* = U_* \cup \cup_{u \in U_*} V N_L(u)$, we have $|U'_* \cap U_j| \leq c_0|U_j|$ for every $j \in [k]$,

then $G \setminus M$ contains a copy of $L$, in which for each $j \in [k]$ the vertices of $U_j$ correspond to the vertices of $V_j$, and $u$ corresponds to a vertex of $Z_u$ for every $u \in U_*$. 

Then the next lemma by Keevash claims that given a regular $k$-partite $k$-complex $G$ with sufficient density (the densities must be much greater than the measure of regularity), and a $k$-partite $k$-graph $M$ on the same vertex set which is small relative to $G$, we can delete a small number of vertices to obtain a robustly universal pair.
Lemma 3 (Theorem 6.32 in [18], see also Theorem 3.3 in [19]). Suppose that $0 \leq 1/n \ll \varepsilon \ll c_0 \ll d^* \ll d_a \ll \theta \ll d \ll c \ll 1/k, 1/D$, $G$ is a $k$-partite $k$-complex on $V = V_1 \cup \ldots \cup V_k$ with $|G_{(ij)}| = |V_j| > n$ for every $j \in [k]$, $G$ is $\epsilon$-regular with $d_{ij}(G) \geq d$ and $d(G_{[ij]}) \geq d_a$, and $M \subseteq G_m$ with $|M| \leq \theta|G|$. Then for all $j \in [k]$ we can delete at most $\theta^{1/4}|V_j|$ vertices from each $V_j$ to obtain a $(c,c_0,d^*)$-robustly $D$-universal pair $(G',M')$.

4 Proof of Theorem 1

4.1 Step 1 and the proof of Lemma 1

We will use the following main parameters

$$0 < \frac{1}{n} \ll \varepsilon \ll d^* \ll d_a \ll \frac{1}{a} \ll \nu \ll \theta \ll d \ll c \ll \frac{1}{k}; r,$$

where again $a \ll b$ means that $a$ is sufficiently small compared to $b$. Furthermore, for any of these constants $\alpha$, we might also use $\alpha \ll \alpha' \ll \alpha'' \ll \ldots$ and assume that the above hierarchy extends for these constants as well, say $d'' \ll c \ll c' \ll c'' \ll \frac{1}{k}$, etc.

Consider an $r$-edge colored complete $k$-graph $K_n^{(k)}$. We remove at most $alk$ vertices so that the number of remaining vertices is divisible by $alk$. Let $T = T_1 \cup \ldots \cup T_k$ be an equitable $k$-partition of the remaining vertices and let $|T_i| = n', 1 \leq i \leq k$. Consider the $r$-edge coloring $(H_1, \ldots, H_r)$ of the $k$-partite (crossing) edges only. Let us take the color in this coloring that has the most edges. For simplicity assume that this is $H_1$ and call this color red. We have

$$|H_1| \geq \frac{1}{r}(n')^k. \quad (2)$$

Following the technique in [19] (Section 5.1.1) we apply the strong hypergraph Regularity Lemma (Lemma 2) to $H_1$ with $t = k$ (this is possible as the number of vertices is divisible by $alk$). We get an $a$-bounded $\epsilon$-regular vertex-equitable partition $(k-1)$-complex $P$ on $T$ and a $k$-partite $k$-graph $G$ on $T$ that is $\nu$-close to $H_1$ and perfectly $\epsilon$-regular with respect to $P$.

Let $M = G \setminus H_1$. Thus any edge of $G \setminus M$ is indeed also an edge of $H_1$. Let $V_1, \ldots, V_m$ be the clusters of $P$. So $T = V_1 \cup \ldots \cup V_m$ and $G$ is $m$-partite with vertex classes $V_1 \cup \ldots \cup V_m$. Let $a_1 = m/k$ and $n_1 = |V_i| = n'/a_1$ ($P$ is equitable). We have $a_1 \leq a$, as $P$ is $a$-bounded.

We define the reduced $k$-graph $R$: the vertices of $R$ correspond to the clusters and a $k$-tuple $S$ of vertices of $R$ corresponds to the $k$-partite union $S' = \cup_{i \in S} V_i$ of clusters. The edges of $R$ are those $S \in \binom{[m]}{k}$ for which $G$ has high density and $M$ has
low density, more precisely \( G[S'] \) has density at least \( c'' \) (i.e. \( |G[S']| \geq c''|K_S(S')| \)) and \( M[S'] \) has density at most \( \nu^{1/2} \) (i.e. \( |M[S']| \leq \nu^{1/2}|K_S(S')| \)).

Consider an edge \( S \in R \) and \( S' = \cup_{i \in S} V_i \). The cells of \( P \) induce a partition \( C^{S_1}, \ldots, C^{S_m} \) of the edges of \( K_S(S') \), where \( m \leq a_k \). We would like to select a cell \( C^{S_i} \) with nice properties. We can have at most \( c''|K_S(S')|/3 \) edges of \( K_S(S') \) within cells \( C^{S_i} \) with \(|C^{S_i}| \leq c''|K_S(S')|/(3a_k) \). Furthermore, we can have at most \( \nu^{1/4}|K_S(S')| \) edges of \( K_S(S') \) within cells \( C^{S_i} \) with \(|M \cap C^{S_i}| \geq \nu^{1/4}|C^{S_i}| \) since \(|M[S']| \leq \nu^{1/2}|K_S(S')| \). This and the fact that \(|G[S']| \geq c''|K_S(S')| \) implies that at least \( c''|K_S(S')|/2 \) edges of \( G[S'] \) lie in cells \( C^{S_i} \) with \(|C^{S_i}| > c''|K_S(S')|/(3a_k) \) and \(|M \cap C^{S_i}| < \nu^{1/4}|C^{S_i}| \). Thus there must exist such a set \( C^{S_i} \) that also satisfies \(|G \cap C^{S_i}| > c''|C^{S_i}|/2 \). Fix one such a choice for \( C^{S_i} \) and denote it by \( C^S \). Let \( G^S \) be the \( k \)-partite \( k \)-complex on the vertex set \( S' \) consisting of \( G \cap C^S \) and the cells of \( P \) that underlie \( C^S \). We also define the \( k \)-partite \( k \)-complex \( M^S = G^S \cap M \) on the vertex set \( S' \). Then the \( k \)-partite \( k \)-complex \( G^S \) has the following properties:

(A1) \( G^S \) is \( \epsilon \)-regular,

(A2) \( G^S \) has \( k \)-th level relative density \( d_{[k]}(G^S) > c''/2(\gg d) \),

(A3) \( G^S \) has absolute density \( d(G^S) \geq (c'')^2/6a_k(\gg d_k) \),

(A4) \( M^S \) satisfies \( |M^S| < 2\nu^{1/4}|G^S|/c''(\ll \theta|G^S|) \),

(A5) \( (G^S)_{(i)} = V_i \) for any \( i \in S \).

Indeed, (A1) follows from the fact that \( G \) is perfectly \( \epsilon \)-regular with respect to \( P \). To see (A2), note that \( (G^S)_{[k]} = C^S \) and so

\[
d_{[k]}(G^S) = \frac{|G^S_{[k]}|}{|(G^S)_{[k]}|^*} = \frac{|G^S \cap C^S|}{|C^S|} > c''/2
\]

by our choice of \( C^S \). Similarly, (A3) follows from our choice of \( C^S \) since

\[
d(G^S) = \frac{|G^S_{[k]}|}{|K_S(S')|} = \frac{|G^S \cap C^S|}{|C^S|} \cdot \frac{|C^S|}{|K_S(S')|} > \frac{(c'')^2}{6a_k}.
\]

Finally, (A4) holds since \(|G^S| > |G \cap C^S| > c''|C^S|/2 \) and \(|M^S| \leq |M \cap C^S| < \nu^{1/4}|C^S| \) and (A5) follows from the construction.

As it is usual in this type of proofs we have to show that the reduced \( k \)-graph satisfies similar density conditions as the original \( k \)-graph \( H_1 \) (see (2)):

\[
|R| \geq \left( \frac{1}{r} - 2c'' \right) a_k^k.
\]
For this purpose first we estimate how many edges of $H_1$ do not belong to $G[S']$ for some edge $S \in R$. There are three possible reasons why an edge $e \in H_1$ does not belong to such a restriction:

(i) $e$ is not an edge of $G$. There are at most $\nu(n')^k$ edges of this type since $H_1$ and $G$ are $\nu$-close.

(ii) $e \in G$ contains vertices from $V_{i_1}, \ldots, V_{i_k}$ such that the restriction of $M$ to $S' = \bigcup_{i \in S} V_i$ satisfies $|M[S']| > \nu^{1/2}|K_S(S')| = \nu^{1/2}n_1^k$, where $S = \{i_1, \ldots, i_k\}$ (note that since $G$ and thus $M$ is $m$-partite, $i_1, \ldots, i_k$ are all distinct). There are at most $\nu^{1/2}(n')^k$ edges of this type since $H_1$ and $G$ are $\nu$-close.

(iii) $e \in G$ contains vertices from $V_{i_1}, \ldots, V_{i_k}$ such that the restriction of $G$ to $S' = \bigcup_{i \in S} V_i$ has density less than $\sigma''$. There are at most $\sigma''(n')^k$ edges of this type.

Therefore using $\nu \ll c$ there are fewer than $2\sigma''(n')^k$ edges of $H_1$ that do not belong to the restriction of $G$ to $S'$ for some $S \in R$. Then using (2) and $n' = a_1n_1$ we get

$$\frac{1}{r}(n')^k \leq |H_1| \leq 2\sigma''(n')^k + |R|n_1^k,$$

from which we get (3).

Next we will use the following simple lemma from [26].

**Lemma 4 (Claim 4.1 in [26]).** Given $c > 0$ and $k \geq 2$, every $k$-partite $k$-graph with at most $m$ vertices in each partition set and with at least $cm^k$ edges contains a tight path on at least $cm$ vertices.

Applying this lemma for the $k$-partite $k$-graph $R$ using (3) we can find a tight path on at least $(\frac{1}{r} - 2\sigma'')a_1$ vertices in $R$. By taking consecutive disjoint edges along this path until we can get a self-connected matching $CM$ with $t$ edges in $R$, such that the number of vertices covered is

$$kt \geq (\frac{1}{r} - 3\sigma'')a_1 = (\frac{1}{r} - 3\sigma'')m \frac{m}{k},$$

as desired in Step 1.

Denote the $i$-th edge of $CM$ by $S(i)$, the corresponding clusters by $\{X_{i1}', \ldots, X_{ik}'\}$, $X_i' = \bigcup_{j=1}^k X_{ij}'$, $G_i' = G^{S(i)}$ and $M_i' = M^{S(i)}$ (where the $k$-partite $k$-complex $G^{S(i)}$ and the $k$-partite $k$-graph $M^{S(i)}$ were defined above). We have

$$d(H_1[X_i']) \geq \sigma''/2 \text{ for all } i \in [t].$$

Indeed, since $S(i) \in R$, $G[X_i']$ has absolute density at least $\sigma''$ and $M[X_i']$ has density at most $\nu^{1/2}$. Then $G \setminus M \subseteq H_1$ and $\nu \ll \sigma''$ imply (5). Furthermore, (A1)-(A5)
imply that we have the following situation: $G'_i$ is an $\epsilon$-regular $k$-partite $k$-complex on the vertex set $X'_i$, with absolute density $d(G''_i) \geq (c''\epsilon)^2/6a^k \gg d_a$, relative density $d_{|k|}(G'_i) > c''/2 \gg d$, $(G'_i)_{(ij)} = X'_{i,j}$ for any $j \in S(i)$ and $|M'_i| < 2n^{1/4}|G'_i|/c'' \ll \theta|G''_i|$. Thus by applying the hypergraph Blow-up Lemma of Keevash (Lemma 3), we can delete at most $\theta'|X'_i|j$ vertices from each $X'_{i,j}$ to obtain a $(c,\epsilon',d')$-robustly $2^k$-universal pair. We delete an additional at most $k - 2$ vertices to make sure that the total number of remaining vertices is divisible by $k - 1$. Let $X_{i,j} \subseteq X'_{i,j}$ and $X_i = \bigcup_{j=1}^{t} X_{i,j}$ denote the remaining vertices and $G_i = G'_i|X_i]$, $M_i = M'_i|X_i]$. The deleted vertices are added back to the remainder of the hypergraph. (Note that here the situation is somewhat simpler than in [19], where they had to include all the vertices on the loose cycle.) The following claim is essentially proved in [19].

Lemma 5. There is a loose cycle in $H_1$ (i.e. a red loose cycle) spanning all the vertices in $\bigcup_{i=1}^{t} X_i$.

Furthermore, this fact will remain true even after the “careful” deletion of up to $n_1/4$ vertices from each cluster. For the sake of completeness we present a sketch of the proof, the missing details can be found in [19] (Sections 5.2-5.3). Then using (4) an immediate corollary is the following result.

Corollary 1. If the edges of the complete $k$-uniform hypergraph $K^{(k)}_{n}$ are colored with $r$ colors then there is a monochromatic loose cycle covering at least $\left(\frac{1}{r} - 4c''\right)\frac{n}{k} \geq \frac{n}{2rk}$ vertices.

This proves Lemma 1 and will be used repeatedly in Step 2.

Sketch of the proof of Lemma 5: The idea is that in each $G_i \setminus M_i$ (and thus in $H_1$) we can find a spanning loose path using the fact that the pair $(G_i, M_i)$ is robustly universal (assuming that $X_i \equiv 1 \bmod (k - 1)$). Then we have to join up all these loose paths we find. However, we will construct these short connecting loose paths first in such a way that the divisibility problems are dealt with. Recall that the edges of $CM$ in $R$ are denoted by $S(1), \ldots, S(t)$. First we find a connecting edge $e'_i$ in $R$ between $S(i)$ and $S(i + 1)$ for $1 \leq i \leq t - 1$: $e'_i$ contains the last $\lfloor k/2 \rfloor$ clusters from $S(i)$ (on the underlying tight path) and the first $\lceil k/2 \rceil$ clusters from $S(i + 1)$ (note that this must be an edge of $R$ as the underlying path is a tight path). We define the supplementary hypergraph $R^*$ on $[t]$ as in [19], then the above $e'_i$ edges translate into a graph path $W = e_1, \ldots, e_{t-1}$, where $e_i = (i, i + 1)$ for all $1 \leq i \leq t - 1$. This corresponds to the connecting walk $W$ in [19] but here the situation is much easier because of the underlying tight path, here all clusters appear exactly once as initial, link or final vertices (see definitions in [19]). The key lemma in [19] is the following lemma which finds a reasonably short connecting loose path in $H_1$ connecting $S(i)$ and $S(i + 1)$ where we may choose (modulo $(k - 1)$) how many vertices this path uses
from $X_i$ and $X_{i+1}$. Furthermore the path avoids a number of forbidden vertices to make sure that these connecting loose paths are disjoint.

**Lemma 6 (Lemma 5.2 in [19]).** Suppose that $e_i = (i, i + 1) \in R^*$ is given as above and $t_1, t_2$ are integers with $0 \leq t_1, t_2 \leq k - 1$ and $t_1 + t_2 \equiv 1 \mod (k - 1)$. Moreover, suppose that $Z$ is a set of at most $100t^2k^3$ forbidden vertices of $H_1$. Then in the sub-$k$-graph of $H_1$ induced by $X_i \cup X_{i+1}$ we can find a loose path $L$ with the following properties.

- $L$ contains at most $4k^3$ vertices.
- $L$ has an initial vertex $u \in X_i$ and a final vertex $v \in X_{i+1}$.
- $|V(L) \cap X_i| \equiv t_i \mod (k - 1)$ for $i = 1, 2$.
- $L$ contains no forbidden vertices, i.e. $V(L) \cap Z = \emptyset$.
- $u$ lies in at least $|H_1[X_i]|/(2|X_i|)$ edges of $H_1[X_i]$ and $v$ lies in at least $|H_1[X_{i+1}]|/(2|X_{i+1}|)$ edges of $H_1[X_{i+1}]$.

We will also need the notion of a prepath from [19]: given a loose path $L$ in some $k$-graph $K$ with initial vertex $x'$ and final vertex $y'$ and disjoint sets $I, F \subseteq V(K) \setminus V(L)$ of size $k - 2$, $L^* = I \cup F \cup V(L)$ is a prepath. If we can find vertices $x, y \in V(K) \setminus L^*$ such that \{x, x'\} $\cup$ I, \{y, y'\} $\cup$ $F$ $\in$ $K$, then adding $x$ and $y$ to $L^*$ gives a loose path; these $x \in V(K)$ are called possible initial vertices of $L^*$ and these $y \in V(K)$ are called possible final vertices of $L^*$. We can use this idea to connect loose paths together: if $L, L', L''$ are disjoint loose paths, $I, F, x, y$ as above, $x$ is also the final vertex of $L'$ and $y$ is also the initial vertex of $L''$ then $I$ and $F$ together with $L', L, L''$ form a single loose path.

Corresponding to $L_c$ in [19], we start by taking a short loose path $L_0$ in $H_1$ from $X_1$ to $X_i$ such that the initial vertex $x_0$ lies in at least $|H_1[X_i]|/(2|X_i|)$ edges of $H_1[X_i]$ and $y_0$ lies in at least $|H_1[X_i]|/(2|X_i|)$ edges of $H_1[X_i]$. Indeed, by utilizing the underlying tight path in $R$ we can clearly find $L_0$ that uses at most $2t$ vertices. We extend $L_0$ to a prepath $L_0^*$ such that we have many $(\geq c|X_i|)$ possible initial vertices of $L_0^*$ in $X_1$ and we have many $(\geq c|X_i|)$ possible final vertices of $L_0^*$ in $X_i$. Then we apply Lemma 6 to each $e_i = (i, i + 1), 1 \leq i \leq t - 1$ in $W$ in order to find a loose path $L_i$ in $H_1$ connecting $X_i$ and $X_{i+1}$ and which we will extend to a prepath $L_i^*$ with many $(\geq c|X_i|)$ possible initial vertices in $X_i$ and with many $(\geq c|X_{i+1}|)$ possible final vertices of in $X_{i+1}$. Furthermore, we can achieve that if $Y_i = X_i \setminus (L_0^* \cup L_1^* \cup \ldots \cup L_{i-1}^*)$, then $|Y_i| \equiv 1 \mod (k - 1)$. This is clear for $Y_i$ with $1 \leq i \leq t - 1$, but it is also true for $Y_i$ using the fact that the total number of vertices in $CM$ is divisible by $k - 1$. Then we finish by finding spanning loose paths $L'$ in each $H_1[Y_i], 1 \leq i \leq t$ such that the
initial vertex of \( L^i \) is also a possible final vertex for \( L^*_i \) and the final vertex of \( L^i \) is also a possible initial vertex for \( L^*_i \), so we can connect all these loose subpaths into one loose cycle. Indeed, first \( |Y_i| \equiv 1 \mod (k - 1) \) implies that this can be done in the complete \( k \)-partite \( k \)-graph (using Lemma 4.2 in [19], since now the clusters are more or less the same size) and then we can use the fact that the pair \((G_i, M_i)\) is robustly universal, finishing the proof of Lemma 5. Note that the proof will go through even if we “carefully” remove at most \( n_1/4 \) vertices from each cluster in \( CM \), we will return and explain this fact in Step 4.

**4.2 Step 2**

Here we will use Corollary 1 repeatedly. We go back from the reduced hypergraph to the original hypergraph and we remove the vertices in \( CM \), i.e. \( \bigcup_{i=1}^t X_i \). We apply repeatedly Corollary 1 to the \( r \)-colored complete hypergraph induced by \( K_n^{(k)} \setminus \bigcup_{i=1}^t X_i \). This way we choose \( l \) vertex disjoint monochromatic loose cycles in \( K_n^{(k)} \setminus \bigcup_{i=1}^t X_i \). We wish to choose \( l \) such that the remaining set \( B \) of vertices in \( K_n^{(k)} \setminus \bigcup_{i=1}^t X_i \) not covered by these \( l \) cycles has cardinality at most \( n/20(rk)^3 \). Since after \( l \) steps at most

\[
(n - |\bigcup_{i=1}^t X_i|) \left(1 - \frac{1}{2rk}\right)^l
\]

vertices are left uncovered, we have to choose \( l \) to satisfy

\[
(n - |\bigcup_{i=1}^t X_i|) \left(1 - \frac{1}{2rk}\right)^l \leq \frac{n}{20(rk)^3}.
\]

This inequality is certainly true if

\[
\left(1 - \frac{1}{2rk}\right)^l \leq \frac{1}{20(rk)^3},
\]

which in turn is true using \( 1 - x \leq e^{-x} \) if

\[
e^{-x} \leq \frac{1}{20(rk)^3}.
\]

This shows that we can choose \( l \leq 12rk \log (rk) \).

**4.3 Step 3**

The complete bipartite \( k \)-uniform hypergraph \( K^{(2,k-2)}(B, A) \) contains all edges of type \((2, k - 2)\), i.e. edges that contain exactly two vertices from \( B \) and \( k - 2 \) vertices from \( A \) (we assumed \( k \geq 3 \)). The key to this step is the following lemma about \( r \)-colored complete unbalanced bipartite hypergraphs.
Lemma 7. For all integers $r \geq 2$ and $k \geq 3$ there exists a constant $n_0 = n_0(r, k)$ such that if the edges of the complete bipartite hypergraph $K^{(2, k-2)}(B, A)$ are colored with $r$ colors and $n_0 \leq |B| \leq \frac{|A|}{2r(k-2)^2}$, then there are at most $100r \log r$ pairwise disjoint monochromatic loose cycles whose link vertices cover $B$.

This lemma is basically from [12]. For the sake of completeness we present the proof in Section 4.5. Interesting to note that here the number of monochromatic loose cycles needed to cover does not depend on $k$.

We have our self-connected matching $CM$ on the clusters $X_{i,j}$, $1 \leq i \leq t$, $1 \leq j \leq k$. In each cluster $X_{i,j}$ let us take a random subset $X'_{i,j}$ (of size approximately $n_1/4$) by assigning each vertex to the subset with probability $1/4$ independently of all other vertices. Let $X''_{i,j} = X_{i,j} \setminus X'_{i,j}$. A standard Chernoff bound implies that with high probability $|X'_{i,j}| \geq n_1/5$ and $|X''_{i,j}| \geq 3n_1/5$. Let us denote by $A$ the union of all of these subsets $X'_{i,j}$ and recall the definition of $B$ from the previous step. Consider the $r$-colored complete bipartite $k$-uniform hypergraph $K^{(2, k-2)}(B, A)$. We apply Lemma 7 in $K^{(2, k-2)}(B, A)$. The conditions of the lemma are satisfied by the above since (using $|A| \geq n/10rk$)

$$|B| \leq \frac{n}{20(rk)^3} \leq \frac{|A|}{2(rk)^2} \leq \frac{|A|}{2r(k-2)^2}.$$ 

Let us remove the at most $100r \log r$ vertex disjoint monochromatic loose cycles covering $B$ in $K^{(2, k-2)}(B, A)$. In the next step we have to show that the statement of Lemma 5 is still true, i.e. there is a red loose cycle in $H_1$ spanning all the remaining vertices in $\cup_{i=1}^t X_i$.

4.4 Step 4

To verify that there is still a red loose cycle in $H_1$ spanning all the remaining vertices in $\cup_{i=1}^t X_i$ we just have to refer to the fact again that $(G_i, M_i)$ is a $(c, \epsilon', d^*)$-robustly $2k$-universal pair. Then basically we have to check that the remainder in $(G_i, M_i)$ still satisfies (i) in the definition of robust universality. But as in [19] (see (B1)) this follows from the fact that $X''_{i,j}$ was selected randomly and we never delete any vertices from $X''_{i,j}$. Furthermore, using $|X''_{i,j}| \geq 3n_1/5$ the number of remaining vertices in the clusters are still not that far apart so the corresponding complete $k$-partite $k$-graph still contains a spanning loose cycle (see Lemma 4.2 in [19]).

Thus the total number of vertex disjoint monochromatic loose cycles we used to partition the vertex set of $K_n^k$ in the various steps is at most (using $k \geq 3$)

$$12rk \log (rk) + 100r \log r + (k-2) + 1 \leq 12rk \log (rk) + \frac{100}{3} rk \log r + k \leq 50rk \log (rk),$$

finishing the proof. $\square$
4.5 Proof of Lemma 7

We define an $r$-edge-colored complete graph $G$ on the vertex set $B$ as follows: $u, v \in B$ are adjacent by an edge of color $i$ if at least $\frac{|A|}{r-2}$ edges of $K^{(2,k-2)}(B, A)$ are colored with color $i$. Applying the main result of [9] the vertex set of $G$ can be covered by at most $100r \log r$ vertex disjoint monochromatic cycles. We try to make loose cycles from these graph cycles by extending each edge of these cycles with $k-2$ vertices to form a hyperedge of the same color. To achieve this we have to make the extension so that the $(k-2)$-sets of $A$ used are pairwise disjoint. The definition of the edge colors allows to perform this extension greedily. Indeed, assume that we have the required extension for some number of edges and $e$ is the next edge to be extended. Since the cycle partition of $G$ has at most $|B|$ edges, if the $(k-2)$-subsets of $A$ used so far cover $U \subset A$, then $|U| < |B|(k-2)$. However, at least $\frac{|A|}{r}$ edges of $K^{(2,k-2)}(B, A)$ are colored with the color of $e$. Overestimating the number of $(k-2)$-subsets of $A$ intersecting $U$ by $|U|/\binom{k-3}{k-3}$, we get

$$|U|\binom{|A|}{k-3} < |B|(k-2)\binom{|A|}{k-3}.$$ 

We claim that

$$|B|(k-2)\binom{|A|}{k-3} \leq \frac{|A|}{r},$$

i.e. we have an extension that is disjoint from $U$, as desired. Indeed, otherwise we get

$$\frac{|A|}{2r(k-2)} < \frac{|A| - k + 3}{r(k-2)} < |B|(k-2),$$

contradicting the assumptions of the lemma. □

References


