An Improved Bound for Vertex Partitions by Connected Monochromatic K-Regular Graphs

Gábor N. Sárközy,[1] Stanley M. Selkow,[2] and Fei Song[3]

[1] COMPUTER SCIENCE DEPARTMENT,
WORCESTER POLYTECHNIC INSTITUTE,
WORCESTER,MA,USA, 01609
E-mail: gsarkozy@cs.wpi.edu

[2] COMPUTER AND AUTOMATION RESEARCH INSTITUTE,
HUNGARIAN ACADEMY OF SCIENCES,
BUDAPEST, P.O. BOX 63, BUDAPEST, HUNGARY, H-1518

[3] COMPUTER SCIENCE DEPARTMENT,
WORCESTER POLYTECHNIC INSTITUTE,
WORCESTER, MA, USA, 01609
E-mail: sms,fes@cs.wpi.edu

Received February 3, 2011; Revised January 10, 2012

Published online 1 May 2012 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jgt.21661

Abstract: Improving a result of Sárközy and Selkow, we show that for all integers $r, k \geq 2$ there exists a constant $n_0 = n_0(r, k)$ such that if $n \geq n_0$ and the edges of the complete graph K_n are colored with r colors then the vertex set of K_n can be partitioned into at most $100r \log r + 2rk$ vertex disjoint connected monochromatic k-regular subgraphs and vertices. This is close to best possible. © 2012 Wiley Periodicals, Inc. J. Graph Theory 73: 127–145, 2013

Contract grant sponsor: NSF; Contract grant Number: DMS-0968699.

Keywords: *vertex partitions; regulatory lemma*

1. INTRODUCTION

1.1. Vertex Partitions

For any $r, k \geq 2$, let $f(r, k)$ denote the minimum number of connected monochromatic k-regular subgraphs and vertices which suffice to partition the vertices of any complete graph whose edges are r-colored. Throughout this article, single vertices and the empty set are considered to be degenerate k-regular subgraphs. It is not obvious that $f(r, k)$ is a well-defined function. That is, it is not obvious that there is always a partition whose cardinality is independent of the order of the complete graph. Gyárfás in [6] conjectured the existence of $f(r, 2)$, and indeed Erdős, Gyárfás, and Pyber in [5] proved that there exists a constant c such that $f(r, 2) \leq cr^2 \log r$ (throughout this article log denotes natural logarithm). In [5], they conjectured that actually $f(r, 2) = r$.

Conjecture 1 (Erdős, Gyárfás, and Pyber [5]). *In every r-coloring of the edges of a complete graph, its vertex set can be partitioned into r monochromatic cycles.*

For general r, the $O(r^2 \log r)$ bound of Erdős, Gyárfás, and Pyber [5] has been improved to $O(r \log r)$ by Gyárfás, Ruszinkó, Sárközy, and Szemerédi [7]. The case $r = 2$ was conjectured earlier by Lehel and was settled by Łuczak, Rödl and Szemerédi [16] for large n using the Regularity Lemma. Later, Allen [1] gave a proof without the Regularity Lemma and recently Bessy and Thomassé [2] found an elementary argument that works for every n. However, Conjecture 1 remains open for $r \geq 3$.

The generalization of this problem for k-regular graphs (for $k = 2$ we get cycles) was initiated by Pyber, Rödl, and Szemerédi in [17]. They showed that cycles and k-regular graphs for $k > 2$ are fundamentally different because of the corresponding Turán-type results. By the Erdős-Gallai [4] theorem, every graph of order n with more than $p(n-1)/2$ edges contains a cycle of length at least $p + 1$. However, the main result of [17] shows that to find *any* three-regular subgraph, let alone a large one, one needs more than $cn \log \log n$ edges. Furthermore, in [17] (Corollary of Theorem 3 in [17]), the authors also proved that in any r-coloring of the edges of the complete graph K_n, there is a monochromatic k-regular subgraph for any $1 \leq k \leq c_r n$, where c_r is a (very small) constant depending only on r. (Note that in the above corollary, the authors claim this only for *some* $k \geq c_r n$, but the stronger statement is implicit in the proof.)

Sárközy and Selkow in [18] gave the following exponential bound for $f(r, k)$.

Theorem 1. *There exists a constant c such that* $f(r, k) \leq r^{c(r \log r + k)}$, *i.e. for any* $r, k \geq 2$ *and for any r-coloring of the edges of a complete graph, its vertices can be partitioned into at most* $r^{c(r \log r + k)}$ *vertex disjoint connected monochromatic k-regular subgraphs and vertices.*

The main goal of this article is to give a significant improvement on this result for large n.

Theorem 2. *For all integers* $r, k \geq 2$, *there exists a constant* $n_0 = n_0(r, k)$ *such that if* $n \geq n_0$ *and the edges of the complete graph* K_n *are colored with r colors then the vertex set of* K_n *can be partitioned into at most* $100r \log r + 2rk$ *vertex disjoint connected monochromatic k-regular subgraphs and vertices.*

We note that this is not far from being best possible (especially if r is small compared to k), as we have the following lower bound.

Theorem 3.

$$f(r, k) \geq (r-1)(k-1) + 1.$$

One of our tools in the proof of Theorem 2 is a Ramsey-type result for the existence of connected monochromatic k-regular subgraphs that may be of independent interest.

Theorem 4. *For every positive ε and integers $r, k \geq 2$, there exists a constant $n_0 = n_0(\varepsilon, r, k)$ such that for any r-coloring of the edges of a complete graph on $n \geq n_0$ vertices, we can find a connected monochromatic k-regular subgraph spanning at least $(1-\varepsilon)n/r$ vertices.*

Thus, perhaps surprisingly, k-regular graphs for $k > 2$ are not that different from cycles: we can find a connected monochromatic k-regular subgraph almost as large as the largest monochromatic cycle we can guarantee from the Erdős–Gallai theorem.

1.2. Sketch of the Proof of Theorem 2

A matching in a graph G is called *connected* if its edges are all in the same connected component of G. To prove Theorem 2, we apply the edge-colored version of the Regularity Lemma to an r-colored K_n. Then we introduce the so called reduced graph G^R, the graph whose vertices are associated with the clusters and whose edges are associated with dense ε-regular pairs. The edges of the reduced graph will be colored with a color that appears most often on the edges between the two clusters. Then we study large monochromatic connected matchings in the reduced graph. This approach was initiated in [15] and, for example, it played an important role in [9] where the three-color Ramsey numbers of paths for large n have been determined.

We follow the *absorbing* proof technique from [7]. This originated in [5] and is used in many papers in this area (e.g. [7], [10], [18]). We establish the bound on $f(r, k)$ in the following steps.

- Step 1: We find a sufficiently large monochromatic (say red), dense (more precisely half-dense in a sense explained later), connected matching M in G^R.
- Step 2: We remove the vertices of M from G^R and greedily remove a number (depending on r) of vertex disjoint connected monochromatic k-regular subgraphs from the remainder in K_n until the number of leftover vertices is much smaller than the number of vertices associated with M. For this purpose, we will use the Ramsey-type result (Theorem 4) for the existence of connected k-regular subgraphs.
- Step 3: Using a lemma about k-regular subgraph covers of r-colored unbalanced complete bipartite graphs, we combine the leftover vertices with some vertices of the clusters associated with vertices of M. (M absorbs the leftover vertices.)
- Step 4: Finally, after some adjustments through alternating paths with respect to M, we find a red k-regular subgraph spanning the remaining vertices of M.

The proof of Theorem 2 in Section 2 will follow this outline. The proof of Theorem 4 is given where we need it in Section 2.2 Since some steps in the proof are straightforward adaptations of the corresponding steps from [7] to k-regular graphs, at some places we will omit the details. First, we discuss the necessary definitions and tools. Then the easy construction for Theorem 3 is given in Section 3.

1.3. Notation and Definitions

For basic graph concepts see the monograph of Bollobás [3]. Disjoint union of sets will sometimes be denoted by $+$.$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A + B$, and $E \subset A \times B \cdot K_n$ is the complete graph on n vertices, $K(n_1, \dots, n_k)$ is the complete k-partite graph with classes containing $n_1, \dots, n_k$ vertices, P_n (C_n) is the path (cycle) with n vertices. $G(n_1, \dots, n_k)$ is a k-partite graph with classes containing $n_1, \dots, n_k$ vertices. For a graph G and a subset U of its vertices, $G|_U$ is the restriction to U of $G \cdot \Gamma(v)$ is the set of neighbors of $v \in V$. The size of$\Gamma(v)$ is $|\Gamma(v)| = \deg(v) = \deg_G(v)$, the degree of v. For a vertex $v \in V$ and set $U \subseteq V$, we write $N(v, U)$ for the set of neighbors of v in U and $\deg(v, U) = |N(v, U)|$. For a subset $S \subseteq V$, we denote by $N(S, U) = \cap_{v \in S} N(v, U)$, the common neighbors of the vertices of S in U. A graph G on n vertices is γ*-dense* if it has at least $\gamma\binom{n}{2}$ edges. A bipartite graph $G(k, l)$ is γ*-dense* if it contains at least γkl edges. When A, B are disjoint subsets of $V(G)$, we denote by $e_G(A, B)$ the number of edges of G with one end point in A and the other in B. For nonempty A and B,

$$d_G(A, B) = \frac{e_G(A, B)}{|A||B|}$$

is the *density* of the graph between A and B.

Definition 1. *Given a pair (A, B) and a bipartite graph $G = (A, B, E)$, the pair (A, B) is (ε, G)-**regular** if*

$$X \subset A, \quad Y \subset B, \quad |X| > \varepsilon|A|, \quad |Y| > \varepsilon|B| \quad \textit{imply} \quad |d_G(X, Y) - d_G(A, B)| < \varepsilon,$$

*otherwise it is (ε, G)-**irregular**. Furthermore, (A, B, E) is (ε, δ, G)-**super-regular** if it is (ε, G)-regular and*

$$\deg_G(a) > \delta|B| \ \ \forall\, a \in A, \qquad \deg_G(b) > \delta|A| \ \ \forall\, b \in B.$$

Note that we need this nonstandard notation for ε-regularity to make clear which edge-colored graph is being considered.

1.4. Tools

In the proof, an r-color version of the Regularity Lemma and the Blow-up Lemma plays a central role.

Lemma 1 (Regularity Lemma, [19]). *For every positive ε and positive integer l_0, there are positive integers L_0 and n_0 such that for $n \geq n_0$ the following holds. For all graphs $G_1, G_2, \dots, G_r$ with $V(G_1) = V(G_2) = \dots = V(G_r) = V$, $r \geq 2$, $|V| = n$, there is a partition of V into $l + 1$ classes (clusters)*

$$V = V_0 + V_1 + V_2 + \cdots + V_l$$

such that

- $l_0 \leq l \leq L_0$,

- $|V_1| = |V_2| = \cdots = |V_l|$,
- $|V_0| < \varepsilon n$,
- *apart from at most* $\varepsilon\binom{l}{2}$ *exceptional pairs, the pairs* $\{V_i, V_j\}$ *are* (ε, G_s)*-regular for* $s = 1, 2, \ldots, r$.

For an extensive survey on different variants of the Regularity Lemma, see [13]. Next, we will need two well-known properties of ε-regular pairs. The first one claims that every (ε, G)-regular graph contains a large super-regular pair.

Lemma 2 (e.g. Proposition 2.3 in [14]). *Assume that* $G = (A, B, E)$ *is an* (ε, G)*-regular graph of density at least* $\delta > 2\varepsilon$. *We can delete at most* $\varepsilon|A|$ *vertices of A and at most* $\varepsilon|B|$ *vertices of B (denote the resulting graph by* G'*) such that the resulting graph is an* $(\varepsilon/(1-\varepsilon)), \delta - 2\varepsilon, G')$*-super-regular graph.*

The second one claims that in an (ε, G)-regular graph (A, B, E), most sets of vertices in A have a large common neighborhood in any large set $Y \subset B$.

Lemma 3 (e.g. Fact 1.4 in [13]). *Assume that* $G = (A, B, E)$ *is an* (ε, G)*-regular graph of density at least* $\delta > 2\varepsilon$. *If* $Y \subset B$ *and* $(\delta - \varepsilon)^{\ell-1}|Y| > \varepsilon|B|$, $(\ell \geq 1)$*, then*

$$\left|\left\{(x_1, x_2, \ldots, x_\ell) \ : \ x_i \in A, \left|Y \cap \left(\cap_{i=1}^{\ell} N(x_i)\right)\right| < (\delta - \varepsilon)^{\ell}|Y|\right\}\right| \leq \ell\varepsilon|A|^{\ell}.$$

We will also use the following lemma of Sárközy and Selkow [18] (a special case of the Blow-up Lemma, [11], [12]) claiming that a balanced super-regular pair can be spanned by a k-regular subgraph.

Lemma 4 (Lemmas 5, 6 in [18]). *Given an* $\varepsilon > 0$ *and an integer* $k \geq 1$*, if* $G = (A, B, E)$ *is an* (ε, δ, G)*-super-regular pair with* $|A| = |B| = m \geq \frac{k}{\varepsilon^2}$ *and* $\delta > 9\varepsilon$*, then G contains a k-regular spanning subgraph. Furthermore, if* $k \geq 2$ *then G contains a connected k-regular spanning subgraph.*

We will use the above-mentioned Erdős–Gallai [4] theorem.

Lemma 5. *Every graph of order n with more than* $p(n-1)/2$ *edges contains a cycle of length at least* $p + 1$.

Finally, we will also need two lemmas of Gyárfás, Ruszinkó, Sárközy, and Szemerédi from [7].

A matching M in a graph G is called *k-half dense* if one can label its edges as $x_1y_1, \ldots, x_{|M|}y_{|M|}$ so that each vertex of $X = \{x_1, \ldots, x_{|M|}\}$ (called the strong end points) is adjacent in G to at least k vertices of $Y = \{y_1, \ldots, y_{|M|}\}$.

Lemma 6 (Lemma 4 in [7]). *Every graph G of average degree at least* $8k$ *has a nontrivial connected k-half dense matching.*

Here, nontrivial means that the matching contains at least one (and therefore at least k) edges.

Lemma 7 (Lemma 5 in [7]). *Let* $\vec{G} = \vec{G}(V, E)$ *be a directed graph with* $|V| = n$ *sufficiently large and minimum out-degree* $d_+(x) \geq cn$ *for some constant* $0 < c \leq 0.001$. *Then, there are subsets* $X, Y \subseteq V$ *such that*

- $|X|, |Y| \geq cn/2$;

- *From every $x \in X$, there are at least $c^6 n$ internally vertex disjoint paths of length at most c^{-3} to every $y \in Y$ (denoted by $x \hookrightarrow y$).*

2. Proof of Theorem 2

2.1. Step 1

We will assume that n is sufficiently large in terms of k and r and that $k \geq 3$. In fact, for $k = 2$, Theorem 2 follows from the main result of [7] (actually the proof there gives a $98r \log r$ bound). We will use the following main parameters

$$0 < \varepsilon \ll \delta \ll 1, \tag{1}$$

where $a \ll b$ means that a is sufficiently small compared to b. In order to present the results transparently we do not compute the actual dependencies, although it could be done.

Consider an r-edge coloring $(G_1, G_2, \dots, G_r)$ of K_n. Apply the r-color version of the Regularity Lemma (Lemma 1), with ε as in (1) and get a partition of $V(K_n) = V = \cup_{0 \leq i \leq l} V_i$, where $|V_i| = m$, $1 \leq i \leq l$. We define the reduced graph G^R: the vertices of G^R are $p_1, \dots, p_l$, and we have an edge between vertices p_i and p_j if the pair $\{V_i, V_j\}$ is (ε, G_s)-regular for $s = 1, 2, \dots, r$. Thus, we have a one-to-one correspondence $f : p_i \to V_i$ between the vertices of G^R and the clusters of the partition. Then,

$$|E(G^R)| \geq (1 - \varepsilon) \binom{l}{2},$$

and thus G^R is a $(1 - \varepsilon)$-dense graph on l vertices.

Define an edge-coloring $(G_1^R, G_2^R, \dots, G_r^R)$ of G^R by r colors in the following way. The edge $p_i p_j$ is colored with a color s that contains the most edges from $K(V_i, V_j)$, thus clearly $e_{G_s}(V_i, V_j) \geq \frac{1}{r}|V_i||V_j|$. Let us take the color class in this coloring of G^R that has the most edges. For simplicity, assume that this is G_1^R and call this color red. Clearly, we have

$$\left|E\left(G_1^R\right)\right| \geq (1 - \varepsilon)\frac{1}{r} \binom{l}{2}, \tag{2}$$

and thus using (1) the average degree in G_1^R is at least $(1 - \varepsilon)(l - 1)/r \geq l/2r$. Using Lemma 6, we can find a connected $l/16r$-half dense matching M in G_1^R. Say M has size

$$|M| = l_1 \geq \frac{l}{16r}, \tag{3}$$

and the matching $M = \{e_1, e_2, \dots, e_{l_1}\}$ is between the two sets of end points U_1 and U_2, where U_1 contains the strong end points, i.e. the points in U_1 have at least $l/16r$ neighbors in U_2. Furthermore, define $f(e_i) = (V_1^i, V_2^i)$ for $1 \leq i \leq l_1$, where V_1^i is the cluster assigned to the strong end point of e_i, and V_2^i is the cluster assigned to the other end point. Hence, we have our large, red, half-dense, connected matching M as desired in Step 1.

However, we need to do some preparations on the matching M. We will need the following lemma (this will be used later again).

Lemma 8. *Assume that for some positive constant c we find a monochromatic connected matching M (say in G_1^R) covering at least $c|V(G^R)|$ vertices of G^R. Then in the original r-edge colored K_n, we find a connected monochromatic k-regular subgraph in G_1 covering at least $c(1-3\varepsilon)n$ vertices.*

Proof. Note that for $k = 2$, this lemma is well known and has been used extensively (e.g. in [7], [9]). Let us use the same notation as above, the matching $M = \{e_1, e_2, \ldots, e_{l_1}\}$, $f(e_i) = (V_1^i, V_2^i)$ for $1 \le i \le l_1$ and $2l_1 \ge cl$.

First, we make the matching edges super-regular by applying Lemma 2. Then we find connecting paths between the edges of the matching M. Since M is a connected matching in G_1^R we can find a connecting path P_i^R in G_1^R from $f^{-1}(V_2^i)$ to $f^{-1}(V_1^{i+1})$ for every $1 \le i \le l_1$ (for $i = l_1$, we have $i + 1 = 1$). Note that these paths in G_1^R may not be internally vertex disjoint. From these paths P_i^R in G_1^R, we can construct l_1 vertex disjoint connecting (almost) k-regular subgraphs H_i in G_1 connecting V_2^i and V_1^{i+1}. More precisely we construct H_1 with the following simple greedy strategy. Denote $P_1^R = (p_1, \ldots, p_t)$, $2 \le t \le l$, where according to the definition $f(p_1) = V_2^1$ and $f(p_t) = V_1^2$. First, let us take a set C^1 of $2k$ "typical" vertices in $f(p_1) = V_2^1$, more precisely we have $|C^1| = 2k$ and $N_{G_1}(C^1, V_1^1), N_{G_1}(C^1, f(p_2)) \ge (1/r - \varepsilon)^{2k} m$. By (ε, G_1)-regularity and Lemma 3, most $2k$-sets of the vertices in V_2^1 satisfy this. We halve C^1 arbitrarily: $C^1 = C_1^1 \cup C_2^1$, $|C_1^1| = |C_2^1| = k$. Next, we take a set C^2 of $2k$ typical vertices in $N_{G_1}(C^1, f(p_2))$, more precisely we have $|C^2| = 2k$ and $N_{G_1}(C^2, f(p_3)) \ge (1/r - \varepsilon)^{2k} m$. By (ε, G_1)-regularity, most of the vertices satisfy this in $N_{G_1}(C^1, f(p_2))$. Note that between C^1 and C^2, we have a complete bipartite graph $K(2k, 2k)$. Again halve C^2 arbitrarily: $C^2 = C_1^2 \cup C_2^2$, $|C_1^2| = |C_2^2| = k$. We continue in this fashion. Finally, for the last C^t, we take $2k$ typical vertices in $N_{G_1}(C^{t-1}, f(p_t))$.

To define the connecting subgraph H_1, we do the following. First from each $K(2k, 2k)$ between C^i and C^{i+1}, $1 \le i \le t-1$ we take a $\lfloor k/2 \rfloor$-regular subgraph (clearly this can be done). Then if k is odd, we add perfect matchings between C_1^i and C_2^{i+1}, $1 \le i \le t-1$. Again this can be done as $|C_1^i| = |C_2^{i+1}| = k$ and the minimum degree is at least $k/2$, so we can apply the König–Hall theorem. Then for the resulting connecting subgraph H_1, all interior vertices (vertices in $\cup_{i=2}^{t-1} C^i$) have degree k, the degrees in C_1^1 and C_2^t are $\lceil k/2 \rceil$ and the degrees in C_2^1 and C_1^t are $\lfloor k/2 \rfloor$.

Then we move on to the next connecting subgraph H_2. We follow the same greedy procedure, we always take the next subset from the next cluster in P_2^R. However, if the cluster has occurred already on the path P_1^R, then we just have to make sure that we pick vertices that have not been used yet on H_1.

We continue in this fashion and construct the vertex disjoint connecting subgraphs H_i in G_1, $1 \le i \le l_1$. Note that for $k = 3$, these connecting subgraphs may not be connected. However, the final k-regular subgraph will be connected. These will be parts of the final connected k-regular subgraph in G_1. We remove the internal vertices of these subgraphs from G_1. At this point, we might have some discrepancies in the cardinalities of the clusters of a matching edge. We remove some more vertices from some clusters V_j^i of the matching to assure that now we have the same number of vertices left in both clusters of a matching edge. For simplicity, we still keep the notation $f(e_i) = (V_1^i, V_2^i)$ for the modified clusters. Note that from each cluster V_j^i, we removed altogether at most $2\varepsilon m$ vertices.

Finally, by applying Lemma 4, we close the connected k-regular subgraph in G_1 within each super-regular matching edge in such a way that we span all the remaining vertices in (V_1^i, V_2^i). Indeed, let us take a balanced super-regular matching edge. The connected spanning subgraph we want to find in (V_1^i, V_2^i) must have k vertices with degree $\lfloor k/2 \rfloor$, k vertices with degree $\lceil k/2 \rceil$ and all other vertices must have degree k (so here these are the missing degrees in the k-regular subgraph we are constructing). First, remove the vertices with degree $\lfloor k/2 \rfloor$, and by applying Lemma 4 with $\lceil k/2 \rceil$ we find a connected $\lceil k/2 \rceil$-regular subgraph in the remainder (note that $\lceil k/2 \rceil \geq 2$ so we may guarantee a connected subgraph). Remove the edges of this subgraph and those vertices that only need degree $\lceil k/2 \rceil$ and add back the vertices with degree $\lfloor k/2 \rfloor$. Note that since k is a constant these changes do not affect much the super-regular properties. By applying Lemma 4 again with $\lfloor k/2 \rfloor$, we find a $\lfloor k/2 \rfloor$-regular subgraph in the resulting pair (if $k = 3$, we just find a perfect matching) in such a way that we are not using any edges from the bipartite graph between the two sets of vertices with degree $\lfloor k/2 \rfloor$. Again since these sets have a constant size this is not a significant restriction. Note that the reason we need this restriction is to guarantee that the spanning subgraph we are constructing within (V_1^i, V_2^i) is connected. This is indeed the case as now every vertex is connected (even for $k = 3$) to the connected $\lceil k/2 \rceil$-regular subgraph we constructed in the previous step. But then since the spanning subgraph within each (V_1^i, V_2^i) is connected, from the construction it follows that the resulting final subgraph is a connected k-regular subgraph. ■

Returning to Step 1, for our matching $M = \{e_1, e_2, \ldots, e_{l_1}\}$ satisfying (3) we follow the same procedure as in Lemma 8 (so in Lemma 8, we have $c = 1/8r$). However, for technical reasons we postpone the last step, the closing of the k-regular subgraph within each (V_1^i, V_2^i), until the end of Step 4, since in Step 3 we will use some of the vertices in $f(M)$, and we will have to make some adjustments first in Step 4.

2.2. Step 2 and the Proof of Theorem 4

First, we give the proof of Theorem 4. As in Step 1, we apply the Regularity Lemma (but with $\varepsilon/4$ instead of ε) and we define the reduced graph G^R and a coloring in G^R by the most frequent color. Then as in (2), G^R contains a subgraph with at least $(1 - \frac{\varepsilon}{4})\binom{l}{2}/r$ edges of the same color. The Erdős–Gallai extremal theorem for cycles (Lemma 5) assures us that this subgraph contains a cycle of length at least $(1 - \frac{\varepsilon}{4})l/r$. Choosing alternate edges of this cycle yields a monochromatic connected matching and then an application of Lemma 8 yields a connected monochromatic k-regular subgraph covering at least

$$\left(1 - \frac{3\varepsilon}{4}\right)\left(1 - \frac{\varepsilon}{4}\right)\frac{n}{r} \geq (1 - \varepsilon)\frac{n}{r}$$

vertices. This finishes the proof of Theorem 4. ■

Returning to Step 2, we go back from the reduced graph to the original graph and we remove the vertices assigned to the matching M, i.e. $f(M)$. We will apply repeatedly Theorem 4 to the r-colored complete graph induced by $K_n \setminus f(M)$. Indeed, first we apply Theorem 4 to $K_n \setminus f(M)$, then the vertices of the resulting connected monochromatic k-regular subgraph are removed and Theorem 4 is applied again to the remaining graph, etc. This way we choose t vertex disjoint connected monochromatic k-regular subgraphs in $K_n \setminus f(M)$. Define the constant $c = 1/500r$ (thus note $c \leq 0.001$ what is needed in Lemma 7). We wish to choose t such that the remaining set B of vertices in $K_n \setminus f(M)$ not covered by these t cycles has cardinality at most $c^{11}n$. Since after t steps at most

$$(n - |f(M)|)\left(1 - \frac{1-\varepsilon}{r}\right)^t$$

vertices are left uncovered, we have to choose t to satisfy

$$(n - |f(M)|)\left(1 - \frac{1-\varepsilon}{r}\right)^t \leq c^{11}n.$$

This inequality is certainly true if

$$\left(1 - \frac{1-\varepsilon}{r}\right)^t \leq c^{11},$$

which in turn is true using $1 - x \leq e^{-x}$ if

$$e^{-\frac{(1-\varepsilon)t}{r}} \leq c^{11}.$$

This shows that we can choose $t = 12r\lceil \log 500r \rceil$ (assuming that ε is small enough).

We may assume that the number of remaining vertices in B is even by removing one more vertex (a degenerate cycle) if necessary.

2.3. Step 3

This step is similar to the corresponding step in [7]. The key to this step is the following lemma about r-colored complete unbalanced bipartite graphs.

Lemma 9. *There exists a constant n_0 such that the following is true. Assume that the edges of the complete bipartite graph $K(A, B)$ are colored with r colors. If $|A| \geq n_0$, $|B| \leq |A|/2r$, then B can be covered by at most $(k+1)r$ vertex disjoint connected monochromatic k-regular subgraphs.*

The proof of this lemma is postponed until Section 2.5 We have the connected, red matching M of size l_1 between U_1 and U_2. Define the auxiliary directed graph $\vec{G}$ on the vertex set U_1 as follows. We have the directed edge from V_1^i to V_1^j, $1 \leq i, j \leq l_1$ if and only if $(V_1^i, V_2^j) \in G_1^R$. The fact that M is $l/16r$-half dense implies that in $\vec{G}$ for the minimum outdegree we have

$$\min_{x \in U_1} d_+(x) \geq \frac{l}{16r} \geq \frac{|U_1|}{16r}\left(\geq \frac{|U_1|}{500r}\right).$$

Thus, applying Lemma 7 for $\vec{G}$ with $c = \frac{1}{500r}(< 0.001)$, there are subsets $X_1, Y_1 \subset U_1$ such that

- $|X_1|, |Y_1| \geq c|U_1|/2$;
- From every $x \in X_1$, there are at least $c^6|U_1|$ internally vertex disjoint paths of length at most c^{-3} to every $y \in Y_1$ ($x \hookrightarrow y$).

Let X_2, Y_2 denote the set of the other end points of the edges of M incident to X_1, Y_1, respectively. Note that a path in $\vec{G}$ corresponds to an alternating path with respect to M in G_1^R.

In each cluster $V_1^i \in Y_1$, let us consider an arbitrary subset of $c^8|V_1^i|$ vertices. Let us denote by A_1 the union of all of these subsets. Similarly, we denote by A_2 the union of arbitrary subsets of $V_2^j \in X_2$ of size $c^8|V_2^j|$. Then we have

$$|A_1|, |A_2| \geq c^8|f(Y_1)| \geq c^8\frac{c}{2}|f(U_1)| \geq c^8\frac{c}{2}\frac{n}{16r} \geq c^{10}n.$$

Let us divide the remaining vertices in B (B was defined in Step 2) into two equal sets B_1 and B_2. Thus, we have $|B_1|, |B_2| \leq |B| \leq c^{11}n$. We apply Lemma 9 in $K(A_1, B_1)$ and in $K(A_2, B_2)$. The conditions of the lemma are satisfied by the above since $|B_i| \leq |A_i|/2r$ for $i = 1, 2$. Let us remove the at most $(k+1)r$ vertex disjoint connected monochromatic k-regular subgraphs covering B_1 in $K(A_1, B_1)$ and the at most $(k+1)rk$-regular subgraphs covering B_2 in $K(A_2, B_2)$. By doing this, we may create discrepancies in the number of remaining vertices in the two clusters of a matching edge. In the next step, we have to eliminate these discrepancies with the use of the many alternating paths.

2.4. Step 4

Again similar to Step 4 in [7]. By removing the vertex disjoint connected monochromatic k-regular subgraphs covering B_1 in $K(A_1, B_1)$, we have created a "surplus" of $|B_1|$ vertices in the clusters of Y_2 compared to the remaining number of vertices in the corresponding clusters of Y_1. Similarly, by removing the k-regular subgraphs covering B_2 in $K(A_2, B_2)$, we have created a "deficit" of $|B_2|(= |B_1|)$ vertices in the clusters of X_2 compared to the number of vertices in the corresponding clusters of X_1. The natural idea is to "move" the surplus from Y_2 through an alternating path to cover the deficit in X_2.

Denote by s' the minimum of $2k$ and the maximum current surplus in any cluster of Y_2, by d' the minimum of $2k$ and the maximum current deficit in any cluster of X_2. We have $s' > 0$ if and only if $d' > 0$ since the total surplus is always equal to the total deficit. Put $s = \min(s', d')$ and we always move s vertices at a time. Note that if $s < 2k$ then the total surplus (and thus the total deficit) has a constant size. Take an arbitrary cluster $V_2^i \in Y_2$ that has a surplus of s vertices and an arbitrary cluster $V_2^j \in X_2$ that has a deficit of s vertices (note that these must exist by the choice of s).

By the construction, there is an alternating path

$$V_2^j, V_1^j, V_2^{j_1}, V_1^{j_1}, \ldots, V_2^{j_k}, V_1^{j_k}, V_2^i$$

such that $k < c^{-3}$. First, we extend the red (G_1) connecting subgraph H_{j-1} (defined in Step 1) by a four-partite subgraph. The four partite sets (each of size $2k$) in this extension come from the following sets (in this order):

$$V_2^j, V_1^j, V_2^j \cup V_2^{j_1}, V_1^j,$$

where we make sure that the third partite set includes exactly s vertices from $V_2^{j_1}$ (thus, for $s = 2k$, all vertices of the third partite set come from $V_2^{j_1}$). Otherwise the construction of this extension is the same as in the proof of Lemma 8. Similarly, we extend the red connecting subgraphs H_{j_1-1}, H_{j_2-1}, etc., with four-partite subgraphs in such a way that we always use s vertices from $V_2^{j_2}, V_2^{j_3}$, etc. Finally, we extend the red connecting subgraph H_{j_k-1} with a four-partite subgraph in such a way that we use s vertices from V_2^i. The overall effect of these extensions is that we moved the surplus of size s from V_2^i to V_2^j without changing any of the other relative sizes in the edges of the matching. This way we came closer to eliminating the discrepancies, and by iterating this procedure, we can totally eliminate them.

However, we have to pay attention again that during this process we never use up too many vertices from any given cluster. It is not hard to see from the construction that we can guarantee that during the whole process with these extensions we use up at most $7c^2$-fraction of any given cluster. Indeed, the total number of vertices along these extensions is at most

$$3c^{-3}c^{12}n = 3c^9 n. \tag{4}$$

We declare an alternating path forbidden if there is a cluster along the path from which we used up at least a $6c^2$-fraction already with these extensions. Then by (4), the total number of vertex disjoint forbidden alternating paths during the whole process is at most $\frac{c^7}{2}l$, and thus by Lemma 7, we have plenty of nonforbidden alternating paths to choose from between any V_2^j and V_2^i.

Hence after this process, the remaining vertices in a matching edge $f(e_i) = (V_1^i, V_2^i)$ will form a balanced super-regular pair where the parameters are somewhat weaker (say $(2\varepsilon, 1/2r, G_1)$-super-regular). Then as we mentioned at the end of Step 1 we can close the k-regular subgraph to span all the remaining vertices of $f(M)$.

Thus, the total number of vertex disjoint connected monochromatic k-regular subgraphs we used to partition the vertex set of K_n is at most

$$12r\lceil\log(500r)\rceil + 2(k+1)r + 2 \leq 100r\lceil\log r\rceil + 2kr,$$

finishing the proof of Theorem 1. □

2.5. Proof of Lemma 9

Again similar to the corresponding Lemma 6 in [7] but we will use a more recent, improved lemma from [8]. Lemma 9 clearly follows from the following two lemmas (corresponding to Lemmas 7 and 8 in [7]).

Lemma 10. *For every $0 < \varepsilon < 1/2$, there exists a constant $n_0 = n_0(\varepsilon)$ such that the following is true. Assume that the edges of the complete bipartite graph $K(A, B)$ are colored with r colors. If $|A| \geq n_0$, $|B| \leq |A|/2r$, then apart from at most $\varepsilon|B|$ vertices B can be covered by at most r vertex disjoint connected monochromatic k-regular subgraphs.*

Lemma 11. *There exists a constant n_0 such that the following is true. Assume that the edges of the complete bipartite graph $K(A, B)$ are colored with r colors. If $|A| \geq n_0$, $|B| \leq |A|/(8r)^{8(r+1)}$, then B can be covered by at most kr vertex disjoint connected monochromatic k-regular subgraphs.*

Lemma 10 follows easily from Lemma 8 and the following lemma from [8].

Lemma 12 (Theorem 2.2 in [8]). *For some $0 < \varepsilon < 1/9$, assume that the edges of a $(1 - \varepsilon)$-dense bipartite graph $G(A, B)$ are colored with r colors, $|B| \leq 2|A|/3r$. Then there are vertex disjoint monochromatic connected matchings, each of a different color, such that their union covers at least $(1 - \sqrt{\varepsilon})$-fraction of the vertices of B.*

Proof of Lemma 10. Indeed, consider an r-edge coloring $(G_1, G_2, \ldots, G_r)$ of $K(A, B)$. We apply the bipartite r-color version of the Regularity Lemma with $\varepsilon' = \frac{\varepsilon^2}{4}$. By standard arguments, we may assume that for each cluster that is not V_0, all vertices of the cluster belong to the same partite class. Thus, we get a partition $A = V_A^0 + V_A^1 + \cdots + V_A^{l_A}$, $B = V_B^0 + V_B^1 + \cdots + V_B^{l_B}$, where $|V_A^{j_1}| = |V_B^{j_2}| = m$, $1 \le j_1 \le l_A$, $1 \le j_2 \le l_B$ and $|V_A^0| \le \varepsilon|A|$, $V_B^0| \le \varepsilon|B|$. We define the reduced graph G^R: the vertices of G^R are $A^R = \{p_A^{j_1} \mid 1 \le j_1 \le l_A\}$ and $B^R = \{p_B^{j_2} \mid 1 \le j_2 \le l_B\}$, and we have an edge between vertices $p_A^{j_1}$ and $p_B^{j_2}$, if the pair $\{V_A^{j_1}, V_B^{j_2}\}$ is (ε', G_s)-regular for $s = 1, 2, \ldots, r$. Thus, we have a one-to-one correspondence $f : \{p_A^j, p_B^j\} \to \{V_A^j, V_B^j\}$ between the vertices of G^R and the nonexceptional clusters of the partition. Then $G^R = (A^R, B^R)$ is a $(1 - \varepsilon')$-dense bipartite graph. Define an r-edge coloring $(G_1^R, G_2^R, \ldots, G_r^R)$ of G^R in the following way. The edge between the clusters $V_A^{j_1}$ and $V_B^{j_2}$ is colored with a color s that contains the most edges from $K(V_A^{j_1}, V_B^{j_2})$, thus clearly

$$\left|E_{G_s}\left(V_A^{j_1}, V_B^{j_2}\right)\right| \ge \frac{1}{r}\left|V_A^{j_1}\right|\left|V_B^{j_2}\right|.$$

Applying Lemma 12 with ε' to G^R, we get at most r vertex disjoint monochromatic connected matchings that cover at least $(1 - \sqrt{\varepsilon'})$-fraction of the vertices of B^R. Then by applying Lemma 8, we go back to the original graph, from these monochromatic connected matchings, we can construct monochromatic connected k-regular subgraphs that cover at least

$$(1 - \sqrt{\varepsilon'})(1 - 3\varepsilon')|B| \ge \left(1 - \frac{\varepsilon}{2} - 3\frac{\varepsilon^2}{4}\right)|B| \ge (1 - \varepsilon)|B|$$

vertices of B. □

The proof of Lemma 11 will use the following simple lemma (corresponding to Lemma 9 in [7]). Note that this is the only place in the proof of our main theorem where the bound depends on k.

Lemma 13. *Assume that the edges of the complete bipartite graph $K(A, B)$ are colored with r colors. If $(|B| - 1)r^{|B|} < |A|$, then B can be covered by at most $(k - 1)r$ vertex disjoint connected monochromatic k-regular subgraphs.*

Proof of Lemma 13. Denote the vertices of B by $\{b_1, b_2, \ldots, b_{|B|}\}$. To each vertex $v \in A$, we assign a vector $(v_1, v_2, \ldots, v_{|B|})$ of colors, where v_i is the color of the edge (v, b_i). The total number of distinct color vectors possible is $r^{|B|}$. Since we have $|A| > (|B| - 1)r^{|B|}$ vectors, by the pigeon-hole principle, we must have a vector that is repeated at least

$$\frac{|A|}{r^{|B|}} \ge |B|$$

times. In other words, there are at least $|B|$ vertices in A for which the colorings of the edges going to $\{b_1, b_2, \ldots, b_{|B|}\}$ are exactly the same. Now if any (and therefore all) vertex in A has at least k neighbors in B in one color, then we can clearly cover the other end points of these edges in B with one connected k-regular subgraph in this color. However, if the number of edges is less than k for a certain color, then the corresponding end points in B will be isolated vertices in our cover. Thus altogether in the worst case we need $(k - 1)r$ vertex disjoint connected monochromatic k-regular subgraphs to cover B. Note that the bound can be improved to $(k - 1)(r - 1) + 1$ provided $|B| > (k - 1)r$. □

Proof of Lemma 11. This is almost identical to the proof of the corresponding lemma (Lemma 8) in [7]. For the sake of completeness, we repeat the proof here with minor differences. Of course, one difference is that whenever we have a monochromatic connected matching in the reduced graph, we saturate it with a connected k-regular subgraph instead of a cycle by applying Lemma 8. The second difference is again we connect the matching edges in the reduced graph by k-regular subgraphs H_i instead of paths. Finally, we will finish with Lemma 13 resulting in at most $(k-1)r + r = kr$ connected monochromatic k-regular subgraphs in the cover.

We proceed similarly as in the proof of Lemma 10. Consider an r-edge coloring $(G_1, G_2, \ldots, G_r)$ of $K(A, B)$.A is sufficiently large and we may assume that B is sufficiently large as well, since otherwise we are done by Lemma 13. We may assume that

$$|A| = (8r)^{8(r+1)}|B| \tag{5}$$

by keeping a subset of A of this size and deleting the rest. We apply the bipartite r-color version of the Regularity Lemma and similarly as in the proof of Lemma 10 we get the reduced graph $G^R = (A^R, B^R)$ that is an $(1-\varepsilon)$-dense bipartite graph. However, here we will use a multicoloring in G^R. Define an r-edge multicoloring $(G_1^R, G_2^R, \ldots, G_r^R)$ of G^R in the following way. The edge between the clusters $V_A^{j_1}$ and $V_B^{j_2}$ has color s for all s such that

$$\left|E_{G_s}\left(V_A^{j_1}, V_B^{j_2}\right)\right| \geq \delta\left|V_A^{j_1}\right|\left|V_B^{j_2}\right|,$$

where δ is given in (1). We use the following claim from [7].

Claim 1 (Claim 1 in [7]). *There exists a color (say G_1, called red) such that $G_1^R = (A^R, B^R)$ contains a connected (A''^R, B''^R) satisfying the following:*

$$|A''^R| \geq \frac{1}{4(4r)^4} l_A \quad \textit{and} \quad |B''^R| \geq \frac{1}{2(4r)^2} l_B, \tag{6}$$

$$\deg_{G_1^R}\left(p_B^j, A^R\right) \geq \frac{1}{4r} l_A, \quad \forall p_B^j \in B''^R, \tag{7}$$

and

$$\deg_{G_1^R}\left(p_A^j, B''^R\right) \geq \frac{1}{2(4r)^2} l_B, \quad \forall p_A^j \in A''^R. \tag{8}$$

We modify B''^R in the following way. We add any vertex $p_B^j \in (B^R \setminus B''^R)$ to B''^R for which we have

$$\deg_{G_1^R}\left(p_B^j, A''^R\right) \geq \frac{2}{(8r)^{8(r+1)}} l_A \,(\geq l_B)\,. \tag{9}$$

For simplicity, we keep the notation B''^R for the resulting set. Thus, now we may assume that for any $p_B^j \in (B^R \setminus B''^R)$, the inequality (9) does not hold.

Then using (7) and (9) by Hall's theorem, we can find a monochromatic (red) connected matching M covering the vertices B''^R (note that the other end points of the matching edges may not be in A''^R for the original vertices of B''^R).

Denote the found $M = \{e_1, e_2, \ldots, e_{l_1}\}$ and $f(e_i) = (V_A^i, V_B^i)$ for $1 \leq i \leq l_1$ (where $l_1 = |B''^R|$). Similarly, as in the proof of Lemma 8 in Step 1, we make the pairs of clusters belonging to the edges in M super-regular (in red). The exceptional vertices removed from the clusters in B are added to V_B^0. Again, similarly as in the proof of Lemma 8 in Step 1, we find the connecting red subgraphs H_i between the super-regular pairs belonging to edges of M and we make the partite sets equal inside one super-regular pair. However, we postpone the closing of the red connected k-regular subgraph inside each pair of clusters belonging to edges of M. First, we need some technical steps. We go back to the original graph and we consider the set of remaining vertices in B:

$$B_1 = V_B^0 + f(B^R \setminus B''^R).$$

Consider those vertices $v \in B_1$ for which

$$\deg_{G_1}(v, f(A''^R)) \geq \frac{4}{(8r)^{5(r+1)}} |A| \, (\geq |B|) \, . \tag{10}$$

These vertices are removed from B_1 and they will be inserted into the red cycle. (For simplicity, we will keep the notation B_1 for the remaining vertices.) For this purpose, first we need an estimate on the number of vertices satisfying (10). We have $|V_B^0| \leq 2\varepsilon|B|$. Let us consider a $p_B^j \in B^R \setminus B''^R$. Using (1), the definition of the coloring in G^R and the fact that for p_B^j (9) does not hold, the number of red edges between $f(p_B^j)$ and $f(A''^R)$ is at most

$$\frac{2}{(8r)^{8(r+1)}} l_A m^2 + \delta l_A m^2 \leq \left(\frac{2}{(8r)^{8(r+1)}} + \delta \right) |A| m \leq \frac{4}{(8r)^{8(r+1)}} |A| m.$$

This clearly implies that we can have at most $\frac{1}{(8r)^{3(r+1)}} m$ vertices in $f(p_B^j)$ satisfying (10). Thus altogether, the number of vertices satisfying (10) is at most

$$\left|V_B^0\right| + \frac{1}{(8r)^{3(r+1)}} |f(B^R \setminus B''^R)| \leq 2\varepsilon|B| + \frac{1}{(8r)^{3(r+1)}} |B| \leq \frac{2}{(8r)^{3(r+1)}} |B|. \tag{11}$$

To handle the vertices satisfying (10), we are going to extend some of the red connecting subgraphs H_i connecting the edges of M so now they are going to include these vertices. Take the first vertex v satisfying (10). Then clearly there is a cluster $p_A^j \in A''^R$ that is not covered by M for which

$$\deg_{G_1}\big(v, f\big(p_A^j\big)\big) \geq \delta m.$$

Take an arbitrary neighbor of p_A^j in B''^R (there must be many by (8)). Then this neighbor is covered by the matching M, say by the edge e_i, $1 \leq i \leq l_1$ where $f(e_i) = (V_A^i, V_B^i)$. Consider the red connecting subgraph H_{i-1} between $f(e_{i-1})$ and $f(e_i)$ ending with $2k$ vertices in V_A^i. We extend this connecting subgraph H_{i-1} by a six-partite subgraph. The six partite sets (each of size $2k$) in this extension come from the following sets (in this order):

$$V_B^i, f\big(p_A^j\big), \quad v \cup V_B^i, f\big(p_A^j\big), \quad V_B^i, V_A^i,$$

where we make sure that the third partite set includes v. Otherwise the construction of this extension is the same as in the proof of Lemma 8.

We repeat the same procedure for all the other vertices satisfying (10). However, we have to pay attention to several technical details. First, of course in repeating this procedure we always consider the remaining free vertices in each cluster; the internal vertices of the connecting subgraphs are always removed. Second, we make sure that we never use up too many vertices from any cluster. It is not hard to see (using (8), (10), and (11)) that we can guarantee that we use up at most half of the vertices from every cluster. Finally, since we are removing vertices from a pair (V_A^i, V_B^i), we might violate the super-regularity. Note that we never violate the ε-regularity. Therefore, we do the following. After using up, say, $\lfloor \delta^2 m \rfloor$ vertices from a pair (V_A^i, V_B^i), we update the pair as follows. In the pair (V_A^i, V_B^i), we remove all vertices u from V_A^i (and similarly from V_B^i) for which $\deg(u, V_B^i) < (\delta - \varepsilon)|V_B^i|$ (again, we consider only the remaining vertices) and we make the partite sets equal inside one super-regular pair. We add the at most εm vertices removed from V_B^i to V_B^0, check whether they satisfy (10) and if they do, we process them with the above procedure.

This way we can handle all the vertices satisfying (10). Now as in the proof of Lemma 8 in Step 1, we can close the red connected k-regular subgraph inside each super-regular edge of M such that it covers all the remaining vertices in V_B^i.

Remove this red connected k-regular subgraph. Denote the resulting sets by B_1 in B and by A_1 in $f(A''^R)$. Put $A_0 = A$ and $B_0 = B$. By (5), (6), and the fact that the relative proportions in the original graph are almost the same as in the reduced graph, we certainly have

$$|A_1| \geq \frac{1}{(8r)^4}|A_0|. \tag{12}$$

We will apply repeatedly the above procedure in (A_1, B_1). However, we consider only the $(r-1)$-edge multicoloring $(G_2, \ldots, G_r)$ in $K(A_1, B_1)$, the edges in G_1 are deleted. Notice that $|A_1|$ is still sufficiently large. We have three cases depending on the size B_1.

Case 1: $(|B_1| - 1)r^{|B_1|} < |A_1|$.

In this case, we are done by Lemma 13 since we have a covering of B by $(k-1)r+1$ $(\leq kr)$ vertex disjoint connected monochromatic k-regular subgraphs. Thus, we may assume that this case does not hold.

Case 2: $(8r)^{8(r+1)}|B_1| \leq |A_1| \leq (|B_1| - 1)r^{|B_1|}$.

In this case, we may run into the problem that the removed k-regular subgraph may contain almost all vertices of B, i.e. $|B_1| = o(|A_1|)$. In this case, the reduced graph might become empty. To avoid this we keep a subset of A_1 of size $(8r)^{8(r+1)}|B_1|$ (denoted again by A_1) and we delete the rest. By the fact that (10) does not hold we know that before this deletion all vertices in B_1 have small degrees in the color removed (red). But then it may happen that the relative degree (the fraction of the degree and the "new" $|A_1|$) of some vertices in the trimmed B_1 in red will not be small any more, i.e. similarly to (10)

$$\deg_{G_1}(v, A_1) \geq \frac{4}{(8r)^{5(r+1)}}|A_1|. \tag{13}$$

To avoid this, we choose a *random subset* of A_1 of this size (denoted again by A_1 for simplicity). Then the relative degrees of the vertices of B_1 will be roughly the same as before the deletion of the superfluous vertices. To make this precise, we use the following claim from [7].

Claim 2 (Claim 2 in [7]). Let $V_n = \{v_1, \ldots, v_n\}$ with n sufficiently large, $\mathcal{F} = \{S_1, \ldots S_m\}$ with $S_i \subseteq V_n$, $|S_i| \leq cn$ for some constant $0 < c \leq 1$. Then for arbitrary $k > \frac{3}{c}\log m$, there exists a $T \subseteq V_n$ such that

- $|T| = k$,

- $|S_i \cap T| \leq 2ck, \forall i.$

We will apply Claim 2 with the following choices. Let $n = |A_1|, V_n = A_1, m = |B_1|, S_i = N_{G_1}(v_i, A_1)$ for $v_i \in B_1$. Then from (12) and the fact that (10) does not hold it follows that we can select

$$c = \frac{\deg_{G_1}(v_i, A_1)}{|A_1|} < \frac{4}{(8r)^{5(r+1)}} \frac{|A|}{|A_1|} \leq \frac{4(8r)^4}{(8r)^{5(r+1)}}. \tag{14}$$

Clearly, all the conditions of the claim are satisfied so we can select the desired subset of A_1 of size $(8r)^{8(r+1)}|B_1|$.

Case 3: $|A_1| < (8r)^{8(r+1)}|B_1|$.

In this case, we continue with A_1 with no modifications.

Now, we are ready to repeat the above procedure in (A_1, B_1). Note that in Case 3 technically we have a somewhat weaker condition for $|A_1|$ in terms of $|B_1|$ compared to the original $|A_0| = (8r)^{8(r+1)}|B_0|$, but that does not create any difficulties, the procedure still goes through.

We will treat Cases 2 and 3 simultaneously. We apply the bipartite $(r-1)$-color version of the Regularity Lemma for the $(r-1)$-colored bipartite graph between A_1 and B_1. Using the fact that in B_1 (10) does not hold and Claim 2 in Case 2, in both Cases 2 and 3 we still have the $\frac{1}{2r} l_{A_1} l_{B_1}$ lower bound for the number of edges in a color, say G_2^R. In Case 3, the above procedure goes through exactly the same way for (A_1, B_1). Note that in Case 3 in (9) we keep the original

$$\frac{2}{(8r)^{8(r+1)}} l_A$$

lower bound (and we do not use l_{A_1} instead of l_A), and similarly in (10) we keep the

$$\frac{4}{(8r)^{5(r+1)}} |A|$$

lower bound (and we do not use $|A_1|$ instead of $|A|$). However, in Case 2, we replace l_A with l_{A_1} in (9) and $|A|$ with $|A_1|$ in (10). Thus, in both cases similarly to (12) we have

$$|A_2| \geq \frac{1}{(8r)^4} |A_1|,$$

and furthermore if we had Case 3 for (A_1, B_1), then using (12) we have

$$|A_2| \geq \frac{1}{(8r)^4} |A_1| \geq \frac{1}{(8r)^8} |A_0|.$$

However, note that if we had Case 2 for (A_1, B_1), then this last inequality might not hold as the "new" $|A_1|$ might be significantly smaller than $\frac{1}{(8r)^4}|A_0|$.

In general, let us consider the situation after t iterations in (A_t, B_t). Assume that the last time Case 2 occurred was at t' $(\leq t)$. If Case 2 never occurred we put $t' = 0$. The above procedure goes through exactly the same way for (A_t, B_t) but we replace l_A with $l_{A_{t'}}$ in (9) and $|A|$ with $|A_{t'}|$ in (10).

If the procedure terminates after $t (\leq r)$ iterations with no more vertices remaining in B, then we have a cover of B with at most kr vertex disjoint connected monochromatic k-regular subgraphs, as desired. Assuming that the procedure does not terminate after r iterations, so $B_r \neq \emptyset$, we will get a contradiction. Indeed, let us examine the maximum degree to the set A_r in any color for each vertex $v \in B_r$. For G_1 since (10) does not hold, we have

$$\deg_{G_1}(v, A_0) < \frac{4}{(8r)^{5(r+1)}}|A_0|.$$

Then as we saw in (14) in case we have Case 3 for (A_1, B_1) we have

$$\deg_{G_1}(v, A_1) < \frac{4(8r)^4}{(8r)^{5(r+1)}}|A_1|,$$

and in case we have Case 2 for (A_1, B_1) using Claim 2, we have to multiply by an extra factor of 2 to get

$$\deg_{G_1}(v, A_1) < \frac{8(8r)^4}{(8r)^{5(r+1)}}|A_1|.$$

We continue in this fashion, in each iteration we have to multiply the coefficient of $|A_i|$ by a factor of $(8r)^4$ and in addition if it was an iteration where we applied Case 2, then we have to multiply by another factor of 2. Thus, for each vertex $v \in B_r$, we have

$$\deg_{G_1}(v, A_r) < \frac{4(2)^r(8r)^{4r}}{(8r)^{5(r+1)}}|A_r| < \frac{|A_r|}{r}.$$

In this upper bound, we assumed the worst possible case when we have a Case 2 application in each iteration and that is why we get the extra factor of 2^r. Note also that we have this upper bound for the other colors as well, and thus for each vertex $v \in B_r$ and color $1 \leq i \leq r$ we have

$$\deg_{G_i}(v, A_r) < \frac{|A_r|}{r},$$

a contradiction, since in at least one of the colors we must have at least $|A_r|/r$ edges from v to A_r. This finishes the proof of Lemma 11. □

3. Proof of Theorem 3

In this section, we present the easy construction for Theorem 3. Let $A_1, \ldots, A_{r-1}$ be disjoint vertex sets of size $k-1$, and A_r is the set of remaining vertices (assuming $n > (r-1)(k-1)$). The r-coloring is defined in the following way: color 1 is all the edges containing a vertex from A_1, color 2 is all the edges containing a vertex from A_2 and not in color 1, etc. we continue in this fashion. Color $r-1$ is all the edges containing a vertex from A_{r-1} and not in color $1, \ldots, r-2$. Finally, color r is all the edges within A_r.

To show the lower bound let us assume that we have a covering by vertex disjoint connected monochromatic k-regular subgraphs. It is not hard to see that in this covering the vertices in $A_1 \cup \ldots \cup A_{r-1}$ must be isolated vertices. Indeed, to cover any vertex in A_i, $1 \leq i \leq r-1$ by a nontrivial connected monochromatic k-regular subgraph, the only possible color is color i. However, we have to include at least one vertex from the outside of A_i. But then this vertex must have k neighbors in A_i, a contradiction. The vertices in $A_1 \cup \ldots \cup A_{r-1}$ must be indeed isolated vertices. Counting one more subgraph to cover A_r, altogether we need at least $(r-1)(k-1)+1$ connected monochromatic k-regular subgraphs to cover all the vertices. □

REFERENCES

[1] P. Allen, Covering two-edge-coloured complete graphs with two disjoint monochromatic cycles, Combin Probab Comput 17(4) (2008), 471–486.
[2] S. Bessy and S. Thomassé, Partitioning a graph into a cycle and an anticycle, a proof of Lehel's conjecture, J Combin Theory Ser B 100(2) (2010), 176–180.
[3] B. Bollobás, Extremal graph theory, Academic Press, London (1978).
[4] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, Acta Math Sci Hungar 10 (1959), 337–356.
[5] P. Erdős, A. Gyárfás, and L. Pyber, Vertex coverings by monochromatic cycles and trees, J Combin Theory Ser B 51 (1991), 90–95.
[6] A. Gyárfás, Covering complete graphs by monochromatic paths, In Irregularities of Partitions, Algorithms and Combinatorics, Vol. 8, Springer-Verlag, Berlin 1989, pp. 89–91.
[7] A. Gyárfás, M. Ruszinkó, G. N. Sárközy, and E. Szemerédi, An improved bound for the monochromatic cycle partition number, J Combin Theory Ser B 96 (2006), 855–873.
[8] A. Gyárfás, M. Ruszinkó, G. N. Sárközy, and E. Szemerédi, One-sided coverings of colored complete bipartite graphs, *Algorithms and Combinatorics, Topics in Discrete Mathematics*, Volume Dedicated to Jarik Nesetril on the Occasion of his 60th Birthday, 2006, ISBN-10 3-540-33698-2, pp. 133–144.
[9] A. Gyárfás, M. Ruszinkó, G. N. Sárközy, and E. Szemerédi, Three-color Ramsey number for paths, Combinatorica 27(1) (2007), 35–69.
[10] P. Haxell, Partitioning complete bipartite graphs by monochromatic cycles, J Combin Theory Ser B 69 (1997), 210–218.
[11] J. Komlós, G. N. Sárközy, and E. Szemerédi, Blow-up Lemma, Combinatorica 17 (1997), 109–123.
[12] J. Komlós, G. N. Sárközy and E. Szemerédi, An algorithmic version of the Blow-up Lemma, Random Structures and Algorithms 12 (1998), 297–312.
[13] J. Komlós and M. Simonovits, Szemerédi's regularity Lemma and its applications in graph theory, In Combinatorics, Paul Erdős is Eighty (D. Miklós, V.T. Sós, and T. Szőnyi, Eds.), Vol. 2, Bolyai Society Math. Studies, Budapest, 1996, pp. 295–352.
[14] D. Kühn and D. Osthus, Packings in dense regular graphs, Combin Probab Comput 14 (2005), 325–337.
[15] T. Łuczak, $R(C_n, C_n, C_n) \leq (4 + o(1))n$, J Combin Theory Ser. B 75 (1999), 174–187.
[16] T. Łuczak, V. Rödl, and E. Szemerédi, Partitioning two-colored complete graphs into two monochromatic cycles, Probab Combinatorics and Computing 7 (1998), 423–436.
[17] L. Pyber, V. Rödl, and E. Szemerédi, Dense graphs without 3-regular subgraphs, J Combin Theory Ser B 63 (1995), 41–54.

[18] G. N. Sárközy and S. Selkow, Vertex partitions by connected monochromatic *k*-regular graphs, J Combin Theory Ser B 78 (2000), 115–122.
[19] E. Szemerédi, Regular partitions of graphs, Colloques Internationaux C.N.R.S. N^{o} 260—Problèmes Combinatoires et Théorie des Graphes, Orsay, CNRS, Paris 1976, pp. 399–401.