Monochromatic Cycle Partitions of Edge-Colored Graphs

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Abstract: In this article we study the monochromatic cycle partition problem for non-complete graphs. We consider graphs with a given independence number $\alpha(G)=\alpha$. Generalizing a classical conjecture of Erdős, Gyárfás and Pyber, we conjecture that if we r-color the edges of a graph G with $\alpha(G)=\alpha$, then the vertex set of G can be partitioned into at most αr vertex disjoint monochromatic cycles. In the direction of this conjecture we show that under these conditions the vertex set of G can be partitioned into at most $25(\alpha r)^2 \log(\alpha r)$ vertex disjoint monochromatic cycles. © 2010 Wiley Periodicals, Inc. J Graph Theory 66: 57–64, 2010

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1. INTRODUCTION

A. Vertex Partitions by Monochromatic Cycles

Assume that K_n is a complete graph on n vertices whose edges are colored with r colors ($r \geq 1$). How many monochromatic cycles are needed to partition the vertex set of K_n? Throughout the article, single vertices and edges are considered to be cycles. Let $p(r)$ denote the minimum number of monochromatic cycles needed to partition the vertex set of any r-colored K_n. It is not obvious that $p(r)$ is a well-defined function. That is, it is not obvious that there always is a partition whose cardinality is independent of the order of the complete graph. However, in [6] Erdős et al. proved that there exists a constant c such that $p(r) \leq cr^2 \log r$ (throughout this article log denotes natural logarithm). Furthermore, in [6] (see also [8]) the authors conjectured the following.

Conjecture 1. $p(r) = r$.

The special case $r = 2$ of this conjecture was asked earlier by Lehel and for $n \geq n_0$ was first proved by Łuczak, Rödl and Szemerédi [12]. Allen [1] improved on the value of n_0 and finally recently Bessy and Thomassé [3] proved the original conjecture for $r = 2$. For general r the current best bound is due to Gyárfás et al. [9] who proved that for $n \geq n_0(r)$ we have $p(r) \leq 100 r \log r$. Let us also note that the above problem was generalized for complete bipartite graphs (see [6] and Haxell [10]) and for vertex partitions by monochromatic connected k-regular subgraphs (see Sárközy and Selkow [17]).

In this article we study the generalization of this problem for r-edge colorings of graphs which are not necessarily complete. Let $\alpha(G)$ be the *independence number* of G, that is the maximum size of an *independent set*, set of vertices not containing both endpoints of an edge. We consider r-edge colorings of graphs with a given independence number $\alpha(G) = \alpha$. Thus for $\alpha = 1$ we get back the special case of complete graphs. Let $p(\alpha, r)$ denote the minimum number of monochromatic cycles needed to partition the vertex set of any r-colored G with $\alpha(G) = \alpha$. Generalizing Conjecture 1 we conjecture the following.

Conjecture 2. $p(\alpha, r) = \alpha r$.

Note that, if true, the conjecture is best possible. This can be easily seen by taking α cliques of roughly equal size and an r-edge coloring inside each clique which requires at least r monochromatic vertex disjoint cycles to cover (see [6]). The conjecture is known to be true for the special cases $r = 1$ (a theorem of Pósa [14], see also Exercise 8.3 in [11], claiming indeed that the vertices of a graph G can be covered by not more than $\alpha(G)$ vertex disjoint cycles, edges and vertices) and $\alpha = 1, r = 2$ [3]. However, in general at this point we can only prove the following weaker bound on $p(\alpha, r)$.

Theorem 1. *We have $p(\alpha,r) \le 25(\alpha r)^2 \log(\alpha r)$, i.e. if the edges of a graph G with $\alpha(G)=\alpha$ are colored with r colors then the vertex set of G can be partitioned into at most $25(\alpha r)^2 \log(\alpha r)$ vertex disjoint monochromatic cycles.*

B. Sketch of the Proof of Theorem 1

We follow the proof technique from [6], where we have to adapt each step to the case of a non-complete graph G. A *triangle cycle* of length k, denoted by T_k, is a cycle $t_1,t_2,\dots,t_k$ of length k with k additional vertices $a_1,a_2,\dots,a_k$ such that a_i is adjacent to t_i and t_{i+1} for all $1 \le i \le k$ (where we have $t_{k+1}=t_1$). Denote $A=\{a_1,a_2,\dots,a_k\}$. The most important property of a triangle cycle T_k is that after the deletion of any subset of A, we still have a Hamiltonian cycle in the remainder of T_k.

Following the proof technique in [6], we find the cycle cover in the following steps.

- Step 1: We find a sufficiently large monochromatic (say red) triangle cycle T_k in G.
- Step 2: We remove the vertices of T_k from G. We greedily remove a number (depending on α and r) of vertex disjoint monochromatic cycles from the remainder in G until the number of leftover vertices is much smaller than k.
- Step 3: We will decompose the set of leftover vertices B into two classes $B=B' \cup B''$. For B', we will use a lemma about cycle covers of r-colored unbalanced dense bipartite graphs (Lemma 4 that may be of independent interest). Using this lemma we combine the vertices of B' with some vertices of the set A from the triangle cycle T_k.
- Step 4: In B'' we will have $\alpha(G|_{B''}) \le \alpha - 1$, so we can use induction on α.
- Step 5: Finally we find a red cycle spanning the remaining vertices of T_k.

The organization of the article follows this outline, after some tools from Ramsey Theory we discuss each step one by one. Note that several of the steps are very similar to the proof in [6], but for the sake of completeness we repeat the details here again. Since probably our bound is far from best possible anyway, we make no attempt at optimizing the constant coefficient.

2. TOOLS FROM RAMSEY THEORY

For graphs $G_1,G_2,\dots,G_r$, the Ramsey number $R(G_1,G_2,\dots,G_r)$ is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into r disjoint color classes inducing r subgraphs $H_1,H_2,\dots,H_r$, then at least one H_i $(1 \le i \le r)$ has a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by Ramsey's classical result [16]. The number $R(G_1,G_2,\dots,G_r)$ is called the Ramsey number for the graphs $G_1,G_2,\dots,G_r$. There is very little known about $R(G_1,G_2,\dots,G_r)$ for $r \ge 3$ even for very special graphs (see eg. [7, 15]). Here we are interested in the very special case when all but one of the graphs are the same. Let $R_r(G;H)$ denote the Ramsey number $R(G_1,\dots,G_r,H)$, where $G_i=G$ for all $1 \le i \le r$. We are interested in $R_r(K_3;K_m)$. We will use the following basic facts from Ramsey theory.

Lemma 1 (Graham et al. [7], p. 92). $R(K_l,K_m) \le \binom{l+m-2}{l-1}$.

Lemma 2. $R_r(K_3;K_m) \leq r!\,m^{r+1}$.

Proof. We will use induction on r. The statement holds for $r=1$ by using Lemma 1. Assuming the result holds for $r-1$, we prove it for r. Consider an edge-coloring of K_N by $r+1$ colors where there is no monochromatic triangle in the first r colors, and there is no monochromatic K_m in the last color. Consider the graph T consisting of all the edges in the first r colors. We claim that the maximum degree of T is at most

$$r(R_{r-1}(K_3;K_m)-1) < rR_{r-1}(K_3;K_m).$$

Indeed, otherwise there is a vertex v incident with at least $R_{r-1}(K_3;K_m)$ edges of color i for some $1 \leq i \leq r$. The induced subgraph of T on the set of all vertices connected to v by edges of color i cannot contain edges of color i, and thus must contain either a monochromatic triangle of color j for some $1 \leq j \leq r$, $j \neq i$, or an independent set of size m, a contradiction. Thus, indeed the maximum degree of T is less than $D = rR_{r-1}(K_3;K_m)$. But then T contains an independent set of size at least N/D. As this set must be of size smaller than m we conclude that

$$\frac{N}{rR_{r-1}(K_3;K_m)} < m,$$

which, together with the induction hypothesis, implies the desired upper bound. ■

We note that in [2] that there are even stronger estimates in terms of m; however, the dependence in r is not explicit.

3. PROOF OF THEOREM 1

A. Step 1

We may assume that $r \geq 2$, since for $r=1$ Conjecture 2 is a theorem of Pósa [14] (see also Exercise 8.3 in [11]) mentioned earlier. The following lemma gives a lower bound on the size of the largest monochromatic triangle cycle we can find in any r-coloring of a graph G with $\alpha(G)=\alpha$. Let $s = r!\,(\alpha+1)^{r+1}$.

Lemma 3. *If the edges of a graph G with $\alpha(G)=\alpha$ are colored with r colors then there exists a monochromatic triangle cycle T_k with $k \geq n/(12rs^3)$.*

Proof. Let us take a graph G with $\alpha(G)=\alpha$. Using Lemma 2 and the definition of s, in G in any subset of s vertices we must have a monochromatic triangle (since we cannot have an independent set of size $\alpha+1$). But then we have at least

$$\binom{n}{s} \Big/ \binom{n-3}{s-3} \geq n^3/(4s^3)$$

monochromatic triangles in G (using $n \geq 4$ which we may clearly assume). The rest of the proof is identical to the proof of Lemma 2 in [6]. For the sake of completeness we give an overview here. First with a greedy procedure we find a subcollection of these monochromatic triangles of size at least $n^2/(12s^3)$, where any two triangles intersect in at most one vertex. Then we keep only those triangles which are colored with the color used most often (say red); we have at least $n^2/(12rs)^3$ such triangles. By removing vertices successively, we can find a non-empty subset X of vertices of G such that the red triangles inside X have large minimum degree, i.e. at least $n/(12rs)^3$ of them are incident to any vertex of X. Finally we consider a maximal red triangle path in X, and we close it into a triangle cycle. Details can be found in [6]. ■

B. Step 2

Here we will use the easy fact that an r-colored graph G with $\alpha(G)=\alpha$ contains a monochromatic cycle of length at least $n/(2\alpha r)$. Indeed, using the complementary form of a well-known consequence of Turán's theorem (see e.g. inequality (10.1), page 150 in [13]), in the most frequent color of G, the number of edges is at least

$$\frac{n}{2r}\left(\frac{n}{\alpha}-1\right)\geq\frac{n^2}{4\alpha r},$$

(using $n\geq 2\alpha$ which again we may clearly assume) and then we can apply the Erdős–Gallai extremal theorem for cycles (see [5, 4], this theorem claims that if a graph G on n vertices has more than $k(n-1)/2$ edges, then it contains a cycle of length at least $k+1$).

We apply repeatedly the above fact to the r-colored graph induced by $G\backslash T_k$. This way we choose t vertex disjoint monochromatic cycles in $G\backslash T_k$. We wish to choose t such that the remaining set B of vertices in $G\backslash T_k$ not covered by these t cycles has cardinality at most $k/(\alpha r)^3$. Since after t steps at most

$$(n-2k)\left(1-\frac{1}{2\alpha r}\right)^t$$

vertices are left uncovered, we have to choose t to satisfy

$$(n-2k)\left(1-\frac{1}{2\alpha r}\right)^t\leq\frac{k}{(\alpha r)^3}.$$

This inequality is certainly true (using Lemma 3) if

$$\left(1-\frac{1}{2\alpha r}\right)^t\leq\frac{1}{12r(\alpha rs)^3},$$

which in turn is true using $1-x\leq e^{-x}$ if

$$e^{-t/(2\alpha r)}\leq\frac{1}{12r(\alpha rs)^3}.$$

This shows that we can choose $t=\lfloor 8\alpha r\log(\alpha rs)\rfloor$ (using $r\geq 2$).

C. Step 3

In the step we will find the decomposition of the set of remaining vertices $B=B'\cup B''$. We put a vertex $b\in B$ into B' if and only if we have

$$deg(b,A)\geq|A|/\alpha, \tag{1}$$

(where again $A = \{a_1, a_2, \ldots, a_k\}$ is the set of third vertices in the triangle cycle T_k) otherwise b is put into B''.

First we deal with the set B'. The key to this step is the following lemma about r-colored unbalanced dense bipartite graphs that may be interesting on its own. We give the proof of the lemma in Section 3.F.

Lemma 4. *Assume that the edges of the bipartite graph $G(A,B)$ are colored with r colors, where we have $|B| \le |A|/(\alpha r)^3$ and $deg(b,A) \ge |A|/\alpha$ for every $b \in B$. Then B can be covered by at most αr^2 vertex disjoint monochromatic cycles.*

We apply Lemma 4 in the bipartite graph (A, B'). The conditions of the lemma are satisfied by the above since

$$|B'| \le |B| \le \frac{k}{(\alpha r)^3} = \frac{|A|}{(\alpha r)^3}$$

and by (1) we have $deg(b,A) \ge |A|/\alpha$ for all $b \in B'$. Let us remove the at most αr^2 vertex disjoint monochromatic cycles covering B' in (A, B').

D. Step 4

Here we deal with the set B''. The key to this step is the following claim.

Claim 1. *In the graph $G|_{B''}$ we have $\alpha(G|_{B''}) \le \alpha - 1$.*

Indeed, otherwise let us take an independent set $\{b_1, b_2, \ldots, b_\alpha\}$ in $G|_{B''}$. By the definition of B'', we have

$$deg(b_j, A) < |A|/\alpha \quad \text{for every } 1 \le j \le \alpha.$$

But then we can choose a vertex $a \in A$ that is not adjacent to any of the vertices $b_j, 1 \le j \le \alpha$, giving an independent set of size $\alpha + 1$ in G, a contradiction.

But then, we can iterate our whole procedure with $\alpha - 1$ inside $G|_{B''}$.

E. Step 5

After removing the vertex disjoint monochromatic cycles covering B' in (A, B'), we still have a red Hamiltonian cycle in the remainder of the triangle cycle T_k.

Thus, if $g(\alpha, r)$ is our bound on the total number of vertex disjoint monochromatic cycles we used in the partition, we get, after some elementary computation using the definition of s and $r \ge 2$,

$$\begin{aligned} g(\alpha, r) &\le \lfloor 8\alpha r \log(\alpha r s) \rfloor + \alpha r^2 + 1 + g(\alpha - 1, r) \\ &\le 25\alpha r^2 \log(\alpha r) + g(\alpha - 1, r). \end{aligned}$$

Repeating this for all $1 < j < \alpha$ and using the bound $g(1, r) \le 25 r^2 \log r$ from [6] (although the constant is not computed in [6], it is easy to see from the proof that 25 works), we get the bound

$$g(\alpha, r) \le 25(\alpha r)^2 \log(\alpha r),$$

finishing the proof of Theorem 1.

F. Cycle Covers in Dense Unbalanced Bipartite Graphs; Proof of Lemma 4

We follow the proof technique of a similar lemma for complete bipartite graphs from [6] (Lemma 1 in [6]). We partition B into $B_1, B_2, \ldots, B_r$ such that for any $i, 1 \le i \le r$, $b \in B_i$ is adjacent to at least $|A|/(\alpha r)$ vertices of A in color i. For $b \in B$ let $\Gamma_i(b)$ denote the set of neighbors of b in color i. We define the auxiliary graph G_i on the vertex set $B_i, 1 \le i \le r$, in the following way. For $b_1, b_2 \in B_i$, (b_1, b_2) is an edge of G_i if and only if we have

$$|\Gamma_i(b_1) \cap \Gamma_i(b_2)| \ge \frac{|A|}{(\alpha r)^3} (\ge |B|). \tag{2}$$

We have the following claim.

Claim 2. *In the graph G_i we have $\alpha(G_i) \le \alpha r$.*

Indeed, otherwise let us take an independent set $\{b_1, b_2, \ldots, b_{\alpha r+1}\}$ in G_i. By the definition of B_i, we have

$$|\Gamma_i(b_j)| \ge |A|/(\alpha r) \quad \text{for every } 1 \le j \le \alpha r + 1.$$

Using this and (2) we get

$$\begin{aligned}
|A| &\ge \left|\bigcup_{j=1}^{\alpha r+1} \Gamma_i(b_j)\right| \ge \sum_{j=1}^{\alpha r+1} |\Gamma_i(b_j)| - \sum_{1 \le j < k \le \alpha r+1} |\Gamma_i(b_j) \cap \Gamma_i(b_k)| \\
&> |A| \left(\frac{\alpha r+1}{\alpha r} - \binom{\alpha r+1}{2} \frac{1}{(\alpha r)^3} \right) = |A| \left(1 + \frac{1}{\alpha r} - \frac{\alpha r+1}{2(\alpha r)^2} \right) \\
&= |A| \left(1 + \frac{\alpha r - 1}{2(\alpha r)^2} \right) > |A|.
\end{aligned}$$

This contradiction proves the claim. By the theorem of Pósa used at the beginning of the proof, G_i can be partitioned into at most αr vertex disjoint cycles (edges and vertices). Using the definition of G_i and (2), it is easy to find at most αr^2 monochromatic vertex disjoint cycles in (A, B) which cover B.

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