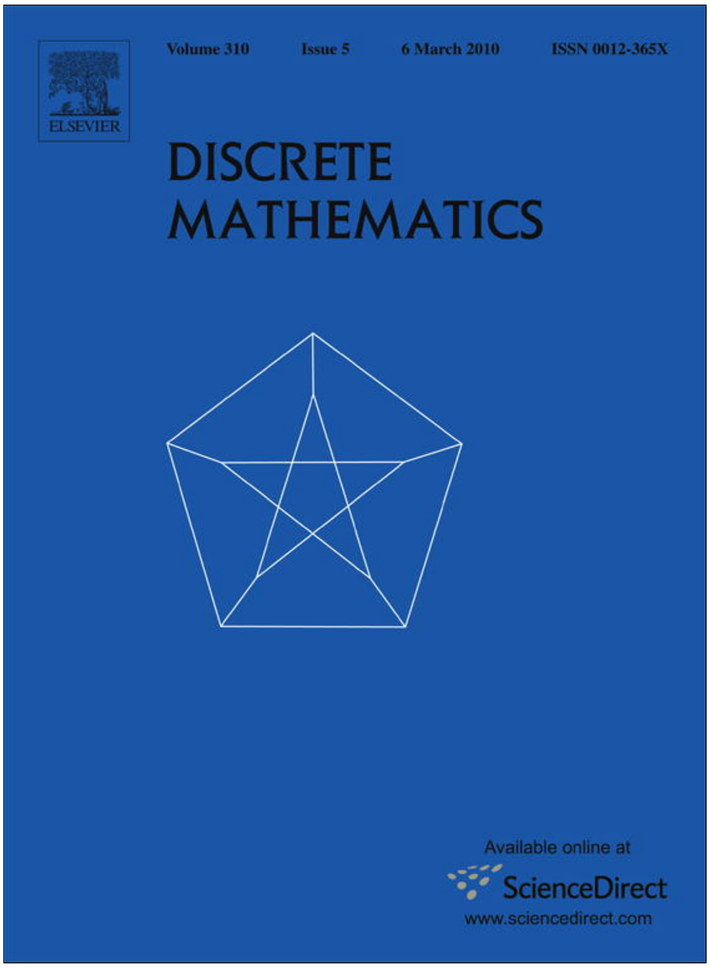
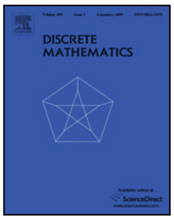

ELSEVIER
Volume 310
Issue 5
6 March 2010
ISSN 0012-365X
DISCRETE MATHEMATICS
Available online at
ScienceDirect
www.sciencedirect.com

Discrete Mathematics 310 (2010) 977–980

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc



Gallai colorings of non-complete graphs

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ARTICLE INFO

Article history:
Received 25 May 2009
Received in revised form 26 September 2009
Accepted 21 October 2009
Available online 8 November 2009

Keywords:
Gallai colorings
Monochromatic connected components

ABSTRACT

Gallai-colorings of complete graphs – edge colorings such that no triangle is colored with three distinct colors – occur in various contexts such as the theory of partially ordered sets (in Gallai's original paper), information theory and the theory of perfect graphs. We extend here Gallai-colorings to non-complete graphs and study the analogue of a basic result – any Gallai-colored complete graph has a monochromatic spanning tree – in this more general setting. We show that edge colorings of a graph H without multicolored triangles contain monochromatic connected subgraphs with at least $(\alpha(H)^2+\alpha(H)-1)^{-1}|V(H)|$ vertices, where $\alpha(H)$ is the independence number of H. In general, we show that if the edges of an r-uniform hypergraph $\mathcal{H}$ are colored so that there is no multicolored copy of a fixed F then there is a monochromatic connected subhypergraph $\mathcal{H}_1 \subseteq \mathcal{H}$ such that $|V(\mathcal{H}_1)| \geq c|V(\mathcal{H})|$ where c depends only on $\mathcal{F}$, r, and $\alpha(\mathcal{H})$.

1. Gallai-colorings of non-complete graphs

Edge colorings of complete graphs in which no triangle is colored with three distinct colors were called Gallai-partitions in [10], Gallai-colorings in [7,6]. More than just the term, the concept occurs again and again in relation to deep structural properties of fundamental objects. An important result in Gallai's original paper [4] – translated to English and endowed by comments in [11] – can be reformulated in terms of Gallai-colorings. Further occurrences are related to generalizations of the perfect graph theorem [2], or applications in information theory [9].

In this paper we start investigating whether Gallai-colorings can be fruitfully extended from complete graphs to arbitrary graphs, i.e. we say that *an edge coloring of a graph G is a Gallai-coloring – or G-coloring – if no triangle of G is colored with three distinct colors.* In particular, every edge coloring of a triangle-free graph is a G-coloring. A less obvious example can be obtained by considering a labeling of the vertices of a graph of order n by $1, 2, \ldots, n$ and for all $1 \leq i < j \leq n$ color the edge ij by color i.

A basic remark of Erdős and Rado states that in any coloring of the edges of a complete graph with two colors there is a monochromatic spanning tree. This remains true for G-colorings of complete graphs as proved in [1], see also [6]. Our starting point is another generalization of the above remark, we state it as Theorem 1. Let $\alpha(H)$ be the *independence number* of H, that is, the maximum size of an *independent set*, set of vertices not containing both endpoints of any edge.

Theorem 1. *If the edges of an arbitrary graph H are colored with two colors, there exists a monochromatic subtree $T \subset H$ with at least $\alpha(H)^{-1}|V(H)|$ vertices.*

We derive Theorem 1 from König's theorem and note that Theorem 1 can be extended for r-colorings as well, with $(r-1)\alpha(H)$ in the role of $\alpha(H)$, this more general form can be obtained from Füredi's result [3] on fractional transversals

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doi:10.1016/j.disc.2009.10.013

(see [5]). Notice that Theorem 1 is sharp if $\alpha(H)$ is a divisor of $|V(H)|$, and simply consider $\alpha(H)$ disjoint monochromatic complete graphs of equal order. Our main result is an analog of Theorem 1 for G-colorings.

Theorem 2. *If the edges of an arbitrary graph H are G-colored, there exists a monochromatic subtree with at least $(\alpha^2(H)+\alpha(H)-1)^{-1}|V(H)|$ vertices and a monochromatic double star with at least $(\alpha^2(H)+\alpha(H)-\frac{2}{3})^{-1}|V(H)|$ vertices.*

In fact, the coefficient of $|V(H)|$ in Theorem 2 is not very far from the truth, showing that the bound of Theorem 1 does not extend to G-colorings. Indeed, consider a triangle-free graph $H = H(\alpha)$ such that $\alpha(H) = \alpha$ and has as many vertices as possible — in other words, H is a so called Ramsey-graph with $R(3, \alpha+1) - 1$ vertices. It is well-known that $|V(H)|$ is "almost" quadratic in α, its order of magnitude is $\frac{\alpha^2}{\log\alpha}$ (see [8]). One can get a trivial G-coloring on H by coloring each edge of H with a different color. Then substituting into each vertex of H a G-colored complete graph K_p, we get a G-colored graph H^* on $p|V(H)|$ vertices, with $\alpha(H^*) = \alpha$ and with largest monochromatic connected subgraph at most $2p$-order of magnitude $\frac{\log\alpha}{\alpha^2}|V(H^*)|$- vertices.

The problem of determining $f(\alpha)$, the largest value such that every G-colored graph H has a monochromatic connected subgraph with at least $f(\alpha(H))|V(H)|$ vertices remains open even for $\alpha = 2$. From Theorem 2 and from the construction above we get

$$\frac{1}{(\alpha^2+\alpha-1)} \le f(\alpha) \le \frac{c\log\alpha}{\alpha^2}$$

where c is a constant, coming from Kim's [8] estimate of $R(3, \alpha+1)$.

For $\alpha = 2$ we have the example above from coloring the edges of a C_5 with five distinct colors, substituting arbitrarily G-colored complete graphs of equal size to the vertices. However, this coloring is not the best for $\alpha = 2$. We may take another Ramsey graph, H_8, the complement of the Wagner graph, its missing edges form an eight cycle $1, 2, \ldots, 8$ plus its main diagonals. (The graph H_8 is a smallest graph with $\alpha = 2$, $\omega = 3$, see for example [13].) The edges of H_8 can be G-colored without a monochromatic connected subgraph on four vertices, by using color i on the edges $(i, i+2)$, $(i, i+5)$ for $i = 1, 2, \ldots, 8$ (modulo 8 arithmetic). Using substitutions into this coloring of H_8 and applying Theorem 2 we have

Corollary 1. $\frac{1}{5} \le f(2) \le \frac{3}{8}$.

Theorem 2 shows that colorings of G without multicolored K_3 have monochromatic connected subgraphs with order proportional to $V(G)$. This property remains true in a much more general setting. Extending the definition from graphs, let $\alpha(\mathcal{H})$ denote the maximum m such that $\mathcal{H}$ contains m vertices that do not contain any edge of $\mathcal{H}$. A hypergraph is *connected* if both parts of every nontrivial 2-partition of its vertex set have nonempty intersection with some edge of the hypergraph.

Theorem 3. *Suppose that the edges of an r-uniform hypergraph $\mathcal{H}$ are colored so that $\mathcal{H}$ does not contain multicolored copies of an r-uniform hypergraph F. Then there is a monochromatic connected subhypergraph $\mathcal{H}_1 \subseteq \mathcal{H}$ such that $|V(\mathcal{H}_1)| \ge c|V(\mathcal{H})|$ where c depends only on F, r, and $\alpha(\mathcal{H})$ (thus does not depend on $\mathcal{H}$).*

2. Proofs

Proof of Theorem 1. Consider a coloring of the edges of H with two colors, say red and blue. Let $\mathcal{H}$ be the hypergraph on vertex set $V(H)$ whose hyperedges are the vertex sets of the connected components (in both colors). Since each vertex of $\mathcal{H}$ is in one red component and in one blue component, the dual of $\mathcal{H}$ is a bipartite graph B. Observe that two vertex-disjoint edges $e, f \in E(B)$ correspond to two vertices $v_e, v_f \in V(H)$ that are not covered by any hyperedge (component) in $\mathcal{H}$, in particular $(v_e, v_f) \notin E(H)$. Therefore the maximum number of pairwise disjoint edges in B, $\nu(B)$, satisfies $\nu(B) \le \alpha(H)$. By König's theorem, the edges of B has a transversal of $\nu(B)$ vertices, i.e. there is $T \subseteq V(B)$ such that $|T| = \nu(B)$ and T intersects all edges of B in at least one vertex. From this it follows that some $t \in T$ is incident to at least $k = \frac{|E(B)|}{\nu(B)}$ edges of B. Thus the hyperedge (component) in $\mathcal{H}$ corresponding to t contains at least

$$k = \frac{|E(B)|}{\nu(B)} = \frac{|V(H)|}{\nu(B)} \ge \frac{|V(H)|}{\alpha(H)}$$

vertices, finishing the proof. □

Proof of Theorem 2. Suppose $f(\alpha)$ is the largest value such that every G-colored graph H has a monochromatic connected subgraph with at least $f(\alpha(H))|V(H)|$ vertices. We estimate $f(\alpha)$ by induction. Clearly $f(1) = 1$ because G-colored complete graphs have monochromatic spanning trees see [1,7,5]. Suppose $\alpha(H) \ge 2$ and consider a G-coloring on H. Let v be an arbitrary vertex, X is the set of vertices nonadjacent to v, A_i is the set of vertices adjacent to v in color i. By induction, the graph $H[X]$ has a monochromatic component C_1 with

$$|V(C_1)| \ge f(\alpha-1)|X|. \tag{1}$$

Set $G = H[\cup_i A_i \cup \{v\}]$ and define the graph G' on the same vertex set as G as follows. For every i, A_i is replaced by a complete graph with each edge colored with color i (all other edges retain their colors). It is enough to find a large monochromatic component in G' because of the following claim.

Claim 1. *Suppose C' is a connected component of G' in color i. Then there is a connected component C of G in color i such that $V(C') = V(C)$.*

Proof. Let C be the subgraph of C' obtained by removing edges of C' in color i that were added when A_i was replaced by a complete graph with each edge colored with color i. Observe that C is connected in color i because for each removed edge xy, the edges xv, vy have color i in G. □

Notice that every edge of G' between A_i and A_j must be colored with either color i or color j because we had a G-coloring on H. Based on this, we can orient the edges of G' so that the edge sets going out from any vertex are monochromatic. To achieve that, for any $1 \le i < j \le n$, orient edges of color i between A_i and A_j from A_i to A_j; similarly, orient edges of color j between A_i and A_j from A_j to A_i. The edges within the A_i's (they are colored with color i) can be oriented arbitrarily. All edges incident to v are oriented into v.

Applying the complementary form of a well-known consequence of Turán's theorem (see for example in [12]) to G'

$$|E(G')| \ge \frac{|V(G')|}{2}\left(\frac{|V(G')|}{\alpha(G')} - 1\right) = M.$$

Thus – by looking at the average outdegree of G' – there exists a vertex $w \in V(G')$ (in fact $w \ne v$ because v has outdegree zero) with outdegree at least

$$\frac{M}{|V(G')|} = \frac{|V(G')|}{2\alpha(G')} - \frac{1}{2}.$$

Since all edges going out of w have the same color, we have a monochromatic component C_2 such that

$$|V(C_2)| \ge \frac{|V(G')|}{2\alpha(G')} \ge \frac{|V(G)|}{2\alpha(G)} \ge \frac{|V(H)| - |X|}{2\alpha(H)}.$$

Combining this estimate with (1), we get a monochromatic component of H with at least

$$\max\left(f(\alpha - 1)|X|, \frac{|V(H)| - |X|}{2\alpha(H)}\right) \tag{2}$$

vertices. Suppose $|X| = \nu|V(H)|$, then $|V(H)| - |X| = (1 - \nu)|V(H)|$ and the two terms in (2) are equal when

$$\nu = \frac{1}{1 + 2\alpha f(\alpha - 1)}$$

and their common value gives the recursion

$$f(\alpha) = \frac{f(\alpha - 1)}{1 + 2\alpha f(\alpha - 1)}$$

and that is equivalent to

$$\frac{1}{f(\alpha)} = \frac{1}{f(\alpha - 1)} + 2\alpha$$

and that easily yields

$$\frac{1}{f(\alpha)} = \alpha^2 + \alpha - 1$$

as desired.

The second statement of the theorem follows easily. For $\alpha = 1$ we have to show that every G-coloring of a complete graph K_n has a monochromatic double star with at least $\frac{3n}{4}$ vertices — this was proved in [6]. This changes only the initial value in the proof above and yields a directed star S of the required size in color i with center $w \in A_i$. Deleting the leaves of S from A_i and adding all edges from v to A_i, S is transformed into the required monochromatic double star.

Proof of Theorem 3. It is enough to prove Theorem 3 for $F = K_m^r$. Assume that the edges of an r-uniform hypergraph $\mathcal{H}$ on n vertices are colored without multicolored F. Set $\alpha = \alpha(\mathcal{H})$ and $t = R(m, \alpha + 1)$, i.e. t is the smallest integer for which every r-uniform hypergraph on t vertices contains either F or $\alpha + 1$ independent vertices. This definition ensures that any set of t vertices of $\mathcal{H}$ contains a copy of F. From this – by an obvious counting argument – we observe that $\mathcal{H}$ contains at least $c_1 n^m$ copies of F where c_1 – as any other c_i later – depends only on m, r, α. Indeed, since any set of t vertices of $\mathcal{H}$ contains a copy of F, there are at least $\frac{\binom{n}{t}}{\binom{n-m}{t-m}} \ge c_1 n^m$ copies of F.

Furthermore – by counting again – there exists $e \in E(\mathcal{H})$ such that e is in at least $c_2 n^{m-r}$ copies of F and the color of e – say red – is repeated in all of these copies. Indeed, assigning two edges of $\mathcal{H}$ with the same color to each copy of F, at least $2c_1 n^m$ edges are assigned, thus some edge e is assigned to at least

$$\frac{2c_1 n^m}{\binom{n}{r}} \geq c_2 n^{m-r}$$

copies of F. We select a second red edge in all of these copies and partition these copies into two parts, $\mathcal{F}_1, \mathcal{F}_2$, the first part contains those copies where the other red edge intersects e, the second part contains those copies where the other red edge does not intersect e. Set $c_3 = \frac{c_2}{2}$, then $\mathcal{F}_i$ must contain at least $c_3 n^{m-r}$ copies of F for $i \in \{1, 2\}$. The union of the second red edges of $\mathcal{F}_i$ forms an r-uniform hypergraph $\mathcal{R}$, its edges cover a (nonempty) set S in $V \setminus e$.

If $i = 1$ then $\mathcal{R} \cup e$ is part of a connected red component and $|S| n^{m-r-1} \geq |\mathcal{F}_1| \geq c_3 n^{m-r}$ thus $|S| \geq c_3 n$. If $i = 2$ then S can be the union of several components of $\mathcal{R}$, say it is the union of components $C_1, C_2, \ldots, C_k$. Let C_1 be the largest component. Then – using that $\sum_{i=1}^{k} \binom{|C_i|}{r}$ is largest (k may vary, $\sum |C_i|$ is fixed and $|C_1| \geq |C_i|$) if all but at most one $|C_i|$ are equal to $|C_1|$-

$$\begin{aligned} c_3 n^{m-r} &\leq |\mathcal{F}_2| \leq \sum_{i=1}^{k} \binom{|C_i|}{r} \binom{n-r}{m-2r} \\ &\leq c_4 n^{m-2r} \sum_{i=1}^{k} \binom{|C_i|}{r} \leq c_4 n^{m-2r} \left\lceil \frac{n}{|C_1|} \right\rceil \binom{|C_1|}{r} \leq c_4 n^{m-2r+1} |C_1|^{r-1}. \end{aligned}$$

Thus we conclude $|C_1| \geq c_5 n$.

In both cases we have a red component in $\mathcal{H}$ with at least $c|V(\mathcal{H})|$ vertices. □

Acknowledgement

Thanks to Miklós Ruszinkó for fruitful conversations on the subject. The first author's research was supported in part by OTKA Grant No. K68322 and the second author's research was supported in part by OTKA Grant No. K68322 and by a János Bolyai Research Scholarship.

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