Monochromatic Hamiltonian t-Tight Berge-Cycles in Hypergraphs

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Abstract: In any r-uniform hypergraph $\mathcal{H}$ for $2 \leq t \leq r$ we define an r-uniform t-tight Berge-cycle of length ℓ, denoted by $C_\ell^{(r,t)}$, as a sequence of distinct vertices $v_1, v_2, \ldots, v_\ell$, such that for each set $(v_i, v_{i+1}, \ldots, v_{i+t-1})$ of t consecutive vertices on the cycle, there is an edge E_i of $\mathcal{H}$ that contains these t vertices and the edges E_i are all distinct for i, $1 \leq i \leq \ell$, where $\ell + j \equiv j$. For $t = 2$ we get the classical Berge-cycle and for $t = r$ we get the so-called tight cycle. In this note we formulate the following conjecture. For

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any fixed $2 \leq c, t \leq r$ satisfying $c + t \leq r + 1$ and sufficiently large n, if we color the edges of $K_n^{(r)}$, the complete r-uniform hypergraph on n vertices, with c colors, then there is a monochromatic Hamiltonian t-tight Berge-cycle. We prove some partial results about this conjecture and we show that if true the conjecture is best possible. J Graph Theory 59: 34–44, 2008

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1. INTRODUCTION

The investigations of Turán type problems for paths and cycles of graphs were started by Erdős and Gallai in [3]. The corresponding Ramsey problems have been looked at some years later first in [6] and then later in [4,5,8,12,14].

There are several possibilities to define paths and cycles in hypergraphs. In this article we address the case of the *Berge-cycle*; probably it is the earliest definition of a cycle in hypergraphs in the book of Berge [1]. Turán type problems for Berge-paths and Berge-cycles of hypergraphs appeared perhaps first in [2]. Other types of hypergraph cycles, *loose* and *tight*, have been studied in [11,13,15]. The investigations of the corresponding Ramsey problems started quite recently with [9,10] where Ramsey numbers of loose and tight cycles have been determined asymptotically for two colors and for 3-uniform hypergraphs.

Let $\mathcal{H}$ be an r-uniform hypergraph (some r-element subsets of a set). Let $K_n^{(r)}$ denote the complete r-uniform hypergraph on n vertices. In any r-uniform hypergraph $\mathcal{H}$ for $2 \leq t \leq r$ we define an r-uniform t-tight Berge-cycle of length ℓ, denoted by $C_\ell^{(r,t)}$, as a sequence of distinct vertices $v_1, v_2, \ldots, v_\ell$, such that for each set $(v_i, v_{i+1}, \ldots, v_{i+t-1})$ of t consecutive vertices on the cycle, there is an edge E_i of $\mathcal{H}$ that contains these t vertices and the edges E_i are all distinct for i, $1 \leq i \leq \ell$, where $\ell + j \equiv j$. We will denote by $E(C_\ell^{(r,t)})$ the set of these edges E_i used on the cycle. For $t = 2$ we get Berge-cycles and for $t = r$ we get the tight cycle. When the uniformity is clearly understood we may simply write $C_\ell^{(t)}$ for $C_\ell^{(r,t)}$ or just C_ℓ. $R_c(C_\ell^{(r,t)})$ will denote the Ramsey number of the r-uniform t-tight ℓ cycle using c colors. A Berge-cycle of length n in a hypergraph of n vertices is called a Hamiltonian Berge-cycle. It is important to remember that, in contrast to the case $r = t = 2$, for $r > t \geq 2$ a Berge-cycle $C_\ell^{(r,t)}$, is not determined uniquely, it can be viewed as an arbitrary choice from many possible cycles with the same triple of parameters.

In this note, continuing the investigations from [7], we study Hamiltonian Berge-cycles in hypergraphs. Thinking in terms of graphs, this task seems quite hopeless, since in many 2-colorings of K_n there are no monochromatic Hamiltonian cycles. For example, if each edge incident to a fixed vertex is red and the other edges are blue, there is no monochromatic Hamiltonian cycle. However, from the nature of Berge-cycles, this example does not carry over to hypergraphs, in this 2-coloring of $K_n^{(3)}$, there is a red Hamiltonian Berge-cycle (for $n \geq 5$).

In [7] monochromatic Hamiltonian (2-tight) Berge-cycles were studied and the following conjecture was formulated. Assume that $r > 1$ is fixed and n is sufficiently large. Then every $(r-1)$-coloring of $K_n^{(r)}$ contains a monochromatic Hamiltonian (2-tight) Berge-cycle. The conjecture was proved for $r = 3$. For general r, the statement was proved for sufficiently large n with $\lfloor \frac{r-1}{2} \rfloor$ colors instead of $r - 1$ colors. In this note we look at monochromatic Hamiltonian t-tight Berge-cycles and we generalize the above conjecture in the following way.

Conjecture 1. *For any fixed $2 \leq c, t \leq r$ satisfying $c + t \leq r + 1$ and sufficiently large n, if we color the edges of $K_n^{(r)}$ with c colors, then there is a monochromatic Hamiltonian t-tight Berge-cycle.*

We will prove that if the conjecture is true it is best possible, since for any values of $2 \leq c, t \leq r$ satisfying $c + t > r + 1$ the statement is not true.

Theorem 2. *For any fixed $2 \leq c,\ t \leq r$ satisfying $c + t > r + 1$ and sufficiently large n, there is a coloring of the edges of $K_n^{(r)}$ with c colors, such that the longest monochromatic t-tight Berge-cycle has length at most $\lceil \frac{t(c-1)n}{t(c-1)+1} \rceil$.*

We know that Conjecture 1 is true for $c = t = 2$ and $r = 3$, see [7]. It has also been proved in [7] that Conjecture 1 is *asymptotically* true for $c = 3$, $t = 2$, and $r = 4$. For the symmetrical case, $c = 2$, $t = 3$, we were able to prove only the following weaker but *sharp* result.

Theorem 3. *For any $n \geq 7$, if the edges of $K_n^{(5)}$ are colored with two colors, then there exists a monochromatic Hamiltonian 3-tight Berge-cycle.*

Note that Conjecture 1 would imply the same statement with $r = 4$ instead of $r = 5$, however, at this point we were unable to prove the statement for $r = 4$.

Similarly as in [7], for general r we were able to obtain only the following weaker result, where essentially we replace the sum $c + t$ with the product ct.

Theorem 4. *For any fixed $2 \leq c, t \leq r$ satisfying $ct + 1 \leq r$ and $n \geq 2(t + 1)rc^2$, if we color the edges of $K_n^{(r)}$ with c colors, then there is a monochromatic Hamiltonian t-tight Berge-cycle.*

In Section 2 we give the simple construction for Theorem 2. In Sections 3 and 4 we present the proofs of Theorems 3 and 4.

2. THE CONSTRUCTION

Proof of Theorem 2. Let $A_1, \ldots, A_{c-1}$ be disjoint vertex sets of size $\lfloor \frac{n}{t(c-1)+1} \rfloor$. The r-edges not containing a vertex from A_1 are colored with color 1. The r-edges that are not colored yet and do not contain a vertex from A_2 are colored with color 2. We continue in this fashion. Finally the r-edges that are not colored yet with colors $1, \ldots, c-2$ and do not contain a vertex from A_{c-1} are colored with color $c - 1$. The r-edges that contain a vertex from all $c - 1$ sets $A_1, \ldots, A_{c-1}$ (if such r-edges exist) get color c. We claim that in this c-coloring of the edges of $K_n^{(r)}$ the longest monochromatic t-tight Berge-cycle has length $\leq \lceil \frac{t(c-1)n}{t(c-1)+1} \rceil$. This is certainly true for Berge-cycles in color i for $1 \leq i \leq c-1$, since the subhypergraph induced by the edges in color i leaves out A_i (a set of size $\lfloor \frac{n}{t(c-1)+1} \rfloor$) completely. Finally, note that in a t-tight Berge-cycle in color c (if such a cycle exists) from t ($> r-c+1$) consecutive vertices on the cycle at least one has to come from $A_1 \cup \cdots \cup A_{c-1}$. Indeed, otherwise the edge containing the t vertices must contain a vertex from each of $A_1, \ldots, A_{c-1}$. Since $t+c-1 > r$, this is a contradiction. Thus the cycle has length at most

$$t(c-1)\left\lfloor \frac{n}{t(c-1)+1} \right\rfloor \leq \frac{t(c-1)n}{t(c-1)+1} \leq \left\lceil \frac{t(c-1)n}{t(c-1)+1} \right\rceil$$

■

3. 3-TIGHT 5-UNIFORM BERGE-CYCLES

Lemma 5. *If the edges of $K_7^{(5)}$ are colored with two colors, there exists a monochromatic Hamiltonian 3-tight Berge-cycle.*

Proof. We first remark that the hypergraph $K_7^{(5)}$ contains 21 edges, that each pair is contained in exactly 10 edges, and each triple is contained in exactly 6 edges.

Let us consider a coloring of the edges of $K_7^{(5)}$ in two colors, blue and red. We will first consider two favorable cases, when the edges containing a pair or a triple of vertices are mostly colored with the same color.

Case 1. Suppose that there exists a pair of vertices (for instance {0, 4}) contained in less than 3 edges of a color (for instance blue); that is it is contained in at least 8 red edges. Without loss of generality, we can assume that if there are blue edges containing {0, 4}, one is (0, 1, 2, 3, 4) and possibly a second one is either (0, 1, 4, 5, 6) or (0, 1, 2, 4, 5).

Let us consider the cycle (0, 6, 2, 3, 4, 5, 1). In Table I, we give a choice of a red edge for each triple of consecutive vertices of this cycle, all distinct.

TABLE I. Choice of a Red Edge for Each Triple for Lemma 5 Case 1

$\{0,6,2\}$:	(0, 2, 4,5,6)
$\{6,2,3\}$:	(0, 2,3,4, 6)
$\{2,3,4\}$:	(0, 2,3,4,5)
$\{3,4,5\}$:	(0, 3,4,5,6)
$\{4,5,1\}$:	(0,1, 3,4,5)
$\{5,1,0\}$:	(0,1,2, 4,5) or (0,1, 4,5,6)
$\{1,0,6\}$:	(0,1, 3,4, 6)

TABLE II. Choice of a Red Edge for Each Triple for Lemma 5 Case 2

$\{0,3,6\}$:	(0,1,2,3, 6)
$\{3,6,2\}$:	(2,3, 6,α,β)
$\{6,2,4\}$:	(0,1,2, 4, 6)
$\{2,4,1\}$:	(0,1,2,3,4)
$\{4,1,5\}$:	(0,1,2, 4,5)
$\{1,5,0\}$:	(0,1,2, 5,6)
$\{5,0,3\}$:	(0,1,2,3, 5)

Case 2. Suppose now that every pair of vertices is contained in at least 3 edges of each color. Suppose that for some triple of vertices, say {0, 1, 2}, all the 6 edges containing it are of the same color, for instance red.

Consider the pair {3, 6}, at least three red edges contains it. One of them is (0, 1, 2, 3, 6), let (3, 6, α, β, γ) be another one. Necessarily, $\{\alpha, \beta, \gamma\} \cap \{0,1,2\} \neq \emptyset$, so we can suppose without loss of generality $\gamma = 2$.

We give in Table II a choice of a red edge for each triple of consecutive vertices for the cycle (0, 3, 6, 2, 4, 1, 5). All these edges are obviously distinct, except perhaps for (2, 3, 6, α, β). Yet this edge may be equal only to (0, 1, 2, 3, 6), and we chose them to be different. So this cycle with this choice of edges forms a red Hamiltonian 3-tight Berge-cycle in $K_7^{(5)}$.

Case 3. Finally, we can assume that every pair of vertices is contained in 3 edges of each color and that every triple of vertices is contained in an edge of each color.

The hypergraph $K_7^{(5)}$ contains 21 edges, so there must be 11 edges of the same color, suppose red. By the pigeonhole principle, we will prove that there must exist a triple that is contained in at least 4 red edges. Each red edge contains exactly $\binom{5}{3} = 10$ distinct triples, this makes at least 110 pairs $\{e, f\}$ such that e is a red edge and f is a triple with $f \subset e$. There are exactly $\binom{7}{3} = 35$ triples, now $\frac{110}{35} > 3$, so there exists a triple that is contained in at least 4 red edges.

Let the triple {0, 1, 2} be contained in at least 4 red edges. It is also contained in a blue edge, suppose (0, 1, 2, 4, 5). If there is a second blue edge containing {0, 1, 2}, *we assume without loss of generality that it is either (0, 1, 2, 3, 6) or (0, 1, 2, 4, 6). Consider the pair* {4, 5}; it is contained in at least 3 red edges: e_1, e_2 and e_3. Since none are equal to (0, 1, 2, 4, 5), they all contain the vertex 3 or 6, maybe both. Moreover, since both triples {3, 4, 5} and {4, 5, 6} are contained in a red edge, then at least one contains 3 and one contains 6. Suppose e_1 contains 3 and e_3 contains 6, e_2 contains either 3 or 6. We consider 3 subcases:

1. If (0, 1, 2, 4, 6) is red:
   In this case, since the edge (0, 1, 2, 3, 4) is also red, we may assume without loss of generality that e_2 contains 6. The edge e_3 contains either 0, 1, or 2;

TABLE III. Choice of a Red Edge for Each Triple for Lemma 5 Case 3

triple :	Subcase 1	Subcase 2	Subcase 3
$\{0,1,2\}$:	(0,1,2, 5,6)	(0,1,2, 5,6)	(0,1,2,3, 5)
$\{1,2,3\}$:	(0,1,2,3, 5)	(0,1,2,3, 5)	(0,1,2,3,4)
$\{2,3,4\}$:	(0,1,2,3,4)	(0,1,2,3,4)	e_1
$\{3,4,5\}$:	e_1	e_1	e_2
$\{4,5,6\}$:	e_2	e_2	e_3
$\{5,6,0\}$:	e_3	e_3	(0,1,2, 5,6)
$\{6,0,1\}$:	(0,1,2, 4, 6)	(0,1,2,3, 6)	(0,1,2,3, 6)

by symmetry, suppose it is 0. We form the cycle (0, 1, 2, 3, 4, 5, 6) with the choice of edges given in Table III, first column.

2. If (0, 1, 2, 4, 6) is blue and e_2 contains 6:
   The edge e_3 necessarily contains a vertex among 0, 1 and 2, suppose it is 0. Then, we form the cycle (0, 1, 2, 3, 4, 5, 6) with the choice of edges given in Table III, second column.
3. If (0, 1, 2, 4, 6) is blue and e_2 contains 3:
   The edge e_1 necessarily contains a vertex among 0, 1 and 2, suppose it is 2. Then, we form the cycle (0, 1, 2, 3, 4, 5, 6) with the choice of edges given in Table III, third column.

Thus in every case, we managed to build a monochromatic Hamiltonian 3-tight Berge-cycle in $K_7^{(5)}$. ■

Proof of Theorem 3. Consider the complete hypergraph $\mathcal{H} = K_n^{(5)}$ whose edges are 2-colored. We will proceed by induction on n, its number of vertices. Lemma 5 establishes the base case for $n = 7$. Let $n \geq 8$. Suppose the result is true for $n - 1$.

Let a be a vertex of $\mathcal{H}$. By the induction hypothesis, the induced subgraph of $\mathcal{H}$ on all its vertices except a has a monochromatic Hamiltonian 5-uniform 3-tight Berge-cycle C. Say its color is *carmine*, the other color being *azure*. Let us name its vertices $\{1, 2, \ldots, n-1\}$ in the order they appear in the cycle.

In the following, we will give a color to any pair $\{x, y\}$ of vertices of $V \setminus \{a\}$, depending on the color of the edges containing x, y, and a. We will say a pair $\{x, y\}$ is *red* if all the edges containing x, y, and a are carmine, except perhaps one. We will say a pair $\{x, y\}$ is *blue* if all the edges containing x, y, and a are azure, except perhaps one. Otherwise, we will say a pair is *green*, meaning at least 2 edges containing x, y and a are carmine and at least 2 are azure.

Remark that if a pair containing x is red, then no pairs containing x can be blue, and vice versa. To prove it, suppose a pair $\{x, y\}$ is red while a pair $\{x, z\}$ is blue.

Take three vertices $u, v, w \notin \{a, x, y, z\}$. Consider the three edges (a, x, y, z, u), (a, x, y, z, v), and (a, x, y, z, w). Two of them have the same color, say carmine, then $\{x, z\}$ cannot be blue, and if the color is azure, $\{x, y\}$ cannot be red.

Suppose first that there exists a $1 \leq i \leq n-1$ such that the pairs $\{i, i+1\}$, $\{i+1, i+2\}$, and $\{i+2, i+3\}$ (with $n-1+j \equiv j$) are green or red. For notation convenience, suppose $i = 1$. We claim that there is a choice of edges such that $(1, 2, a, 3, 4, \ldots, n-1)$ is a 3-tight monochromatic carmine Hamiltonian cycle. Let us define such a choice of edges. For any $3 \leq j \leq n-1$, choose for the set $\{j, j+1, j+2\}$ the corresponding edge in C. Three edges still have to be found, corresponding to the sets $\{1, 2, a\}$, $\{2, a, 3\}$ and $\{a, 3, 4\}$. For these three sets, we will choose edges containing a, that are therefore different from the edges we took before.

Since the pairs $\{1, 2\}$, $\{2, 3\}$, and $\{3, 4\}$ are green or red, there are at least two carmine edges containing each of the sets $\{a, 1, 2\}$, $\{a, 2, 3\}$, and $\{a, 3, 4\}$.

If the edge $(1, 2, 3, 4, a)$ is carmine, take it for the set $\{2, a, 3\}$. Now choose any other carmine edge for $\{1, 2, a\}$ and $\{a, 3, 4\}$. There exist such edges since $\{1, 2\}$ and $\{3, 4\}$ are green or red, and they are distinct since different from $(1, 2, 3, 4, a)$. Otherwise, take any suiting carmine edge for $\{2, a, 3\}$, and different carmine edges for $\{1, 2, a\}$ and $\{a, 3, 4\}$. All these edges exist since $\{1, 2\}$, $\{2, 3\}$, and $\{3, 4\}$ are green or red, and the edge for $\{1, 2, a\}$ and $\{a, 3, 4\}$ are different or it would be $(1, 2, 3, 4, a)$, which is azure.

Now we can suppose that for any $1 \leq i \leq n-1$, $\{i, i+1\}$, $\{i+1, i+2\}$, or $\{i+2, i+3\}$ is blue. Since most edges are now blue, we are tempted to try to form a cycle of color azure. We will still form a carmine cycle in the following case.

Suppose there exists a vertex $1 \leq i \leq n-1$, such that the edges $(a, i, i+1, i+2, i+3)$, $(a, i, i+1, i+2, i+4)$, and $(a, i, i+1, i+2, i+5)$ are carmine. Then to form a carmine cycle, we insert a between $i+1$ and $i+2$. We get the cycle $(1, 2, \ldots, i, i+1, a, i+2, i+3, \ldots, n-1)$. For $\{i, i+1, a\}$, we use the edge $(a, i, i+1, i+2, i+5)$, for $\{i+1, a, i+2\}$, the edge $(a, i, i+1, i+2, i+4)$, for $\{a, i+2, i+3\}$, the edge $(a, i, i+1, i+2, i+3)$, and for all the other triples, we use the corresponding edge of C.

We finally can assume otherwise that for any $1 \leq i \leq n-1$, one of the edges $(a, i, i+1, i+2, i+3)$, $(a, i, i+1, i+2, i+4)$, and $(a, i, i+1, i+2, i+5)$ is azure. Then using this edge for the set $\{i, i+1, i+2\}$, we form an azure cycle C' $\{1, 2, \ldots n\}$ not containing a. All the edges we used are distinct since $n-1 > 6$. Let us choose a blue pair of consecutive vertices in the cycle. Without loss of generality, suppose the pair is $\{2, 3\}$. We will insert the vertex a between 2 and 3 in the cycle C'. Most edges may remain unchanged. For the set $\{1, 2, a\}$, we can use the edge of C' formerly used for $\{1, 2, 3\}$ which contains a by construction of C'. Likewise, we can use for $\{a, 3, 4\}$ the edge of C' formerly used for $\{2, 3, 4\}$. We only have to find an edge for $\{2, a, 3\}$. Since $\{2, 3\}$ is blue, either $(2, a, 3, 5, 6)$ or $(2, a, 3, 5, 7)$ is azure, and they both are distinct from any edge of C'. So we can find among these two an edge for $\{2, a, 3\}$, and we get a monochromatic Hamiltonian 3-tight Berge-cycle. ■

4. PROOF OF THEOREM 4

Proof of Theorem 4. We follow the method of [7]. For the sake of completeness we give the details. We first prove the following lemma.

Lemma 6. *Let k and $t \geq 2$ be fixed positive integers and let $n > 2(t+1)tk$. Then a $(t+1)$-uniform hypergraph $\mathcal{H}$ of order n with at least $\binom{n}{t+1} - kn$ edges has a Hamiltonian t-tight Berge-cycle.*

Proof. By averaging there exists a vertex $x \in V(\mathcal{H})$ contained in at least $\binom{n-1}{t} - (t+1)k$ edges of $\mathcal{H}$. Thus apart from at most $(t+1)k$ exceptional sets all subsets of size t on the remaining $n-1$ vertices form an edge of $\mathcal{H}$ together with x. Let us denote the union of the vertices in the exceptional subsets by U. Thus $|U| \leq (t+1)kt$. Take a cyclic permutation on the remaining vertices where two vertices from U are never neighbors. Since $n > 2(t+1)tk$, this is possible. But then this cyclic permutation is actually a t-tight Berge-cycle, that is, $C_{n-1}^{(t+1,t)}$. Indeed, any set of t consecutive vertices on the cycle contains a non-exceptional vertex and thus it forms an edge with x. Furthermore, since $n > 2(t+1)tk$, there must be two non-exceptional vertices, denoted by x_1 and y_1, that are neighbors on the cycle. Consider the $2t$ consecutive vertices along the cycle that include x_1 and y_1 in the middle, and denote these vertices by $x_t, \ldots, x_1, y_1, \ldots, y_t$. Consider also a vertex z along the cycle that is not among these $2t$ vertices. We claim that x can be inserted between x_1 and y_1 on the cycle and thus giving a Hamiltonian t-tight Berge-cycle in $\mathcal{H}$. Indeed, for those sets of t consecutive vertices which do not include x, we can add x to get the required edge E_i. If a set of t consecutive vertices includes x, then it also must include either x_1 or y_1 (or maybe both), that is, a non-exceptional vertex. But then we can add z to get the required edge. It is easy to check that all the used edges are distinct. ■

For $S \subseteq V(K_n^{(g)})$, $|S| < g$, let $E_S = \{e | e \in E(K_n^{(g)}) \text{ with } S \subseteq e\}$, the set of edges containing S. Thus $|E_S| = \binom{n-|S|}{g-|S|}$. It is enough to prove Theorem 4 for $r = ct+1$. Indeed, for $r > ct+1$, one can have a color transfer by any injection of the $(ct+1)$-element subsets of the n vertices into their r-element supersets ($n \geq 2r$ is ensured). Then Theorem 4 will easily follow from the following stronger theorem where only a fraction of the edges are colored (although with perhaps fewer colors), and we can still manage to find a monochromatic Hamiltonian t-tight Berge-cycle.

Theorem 7. *Let $c, t \geq 2$ and let $n \geq 2(t+1)tc^2$. Furthermore let $S \subseteq V(K_n^{(ct+1)})$ such that S is of order divisible by t (possibly empty) with $|S| \leq (c-1)t$. Set $u = c - \frac{|S|}{t}$ (≥ 1). Color $m \geq \binom{n-|S|}{ct+1-|S|} - (c-u)(n+t) > 0$ edges of E_S with u colors. Then E_S contains a monochromatic Hamiltonian t-tight Berge-cycle.*

Proof. Let $F_S \subseteq E_S$, $|F_S| = m$, be the set of colored edges in E_S. Fix $t \geq 2$. The proof will be by induction on u, $1 \leq u \leq c$. If $u = 1$, then $|S| = (c-1)t$ so that $\binom{n-|S|}{ct+1-|S|} - (c-1)(n+t) = \binom{n-(c-1)t}{t+1} - (c-1)(n+t) \geq \binom{n-(c-1)t}{t+1} - c(n-(c-1)t)$ when $n \geq tc^2$. Define the $(t+1)$-uniform hypergraph $\mathcal{H}_S$ with

$V(\mathcal{H}_S) = V(K_n^{(ct+1)}) \setminus S$ and $E(\mathcal{H}_S) = \{e \setminus S \mid e \in F_S\}$. Therefore since $n-(c-1)t > 2(t+1)tc$ by Lemma 6 $\mathcal{H}_S$ contains a Hamiltonian t-tight Berge-cycle $C_{n-(c-1)t}^{(t+1,t)}$. Then we get the corresponding t-tight Berge-cycle $C_{n-(c-1)t}^{(ct+1,t)}$ in E_S. But each edge of E_S contains S and only $n-(c-1)t$ edges are used on this $C_{n-(c-1)t}^{(ct+1,t)}$ so that it is easy to insert all of S in place of any edge of $C_{n-(c-1)t}^{(ct+1,t)}$ giving the monochromatic $C_n^{(ct+1,t)}$. Indeed, insert all the vertices of S in arbitrary order between two consecutive vertices on the cycle. Consider a set T of t consecutive vertices on the new cycle. If T does not contain a vertex from S, then we can use the edge E_i from $E(C_{n-(c-1)t}^{(ct+1,t)})$. If T does have at least one vertex from S, then it has at most $(t-1)$ vertices outside S, and thus at least $ct+1-|S|-(t-1)=2$ more vertices are "free", so in E_S the number of edges containing T that we can still use (not missing or not used on the cycle yet) is at least

$$\binom{n-|S\cup T|}{2} - (c+1)(n-(c-1)t) \geq \frac{(n-ct)^2}{2} - (c+1)(n-(c-1)t).$$

Thus we can select a distinct edge E_i for each such T if

$$\frac{(n-ct)^2}{2} - (c+1)(n-(c-1)t) \geq ct,$$

which is certainly true for $n \geq 2(t+1)tc^2$.

Therefore assume the theorem holds for $u-1$ colors with $c \geq u \geq 2$ and color the m edges of E_S by u colors, $m \geq \binom{n-|S|}{ct+1-|S|} - (c-u)(n+t) > 0$, $|S| = (c-u)t$. In F_S select a maximum length monochromatic t-tight Berge-cycle. Suppose first that this is $C_\ell^{(ct+1,t)} = (z_1, z_2, \ldots, z_\ell)$ in color 1, with $2t-2 \leq \ell < n$. We will handle the case $\ell < 2t-2$ later. Let $z \in V(K_n^{(ct+1)}) \setminus V(C_\ell^{(ct+1,t)})$. Consider the vertices $\{z_1, z_2, \ldots, z_{2t-2}\}$ (using $2t-2 \leq \ell$) and the t subsets $T_1, \ldots, T_t$ consisting of $t-1$ consecutive vertices in this interval. If for each i, $1 \leq i \leq t$ the set $T_i \cup \{z\}$ is contained in at least t distinct edges in $E_S \setminus E(C_\ell^{(ct+1,t)})$ in color 1, then clearly we could insert z into the cycle between z_{t-1} and z_t, a contradiction. Hence we may assume that for some T_i (say T_1 without loss of generality) apart from at most $(c-u)(n+t)+t$ exceptional edges all edges in $E_{S\cup T_1\cup\{z\}} \setminus E(C_\ell^{(ct+1,t)})$ are in color $2, 3, \ldots, u$.

Assume now the second case, $\ell < 2t-2$. Consider arbitrary vertices $\{z_1, z_2, \ldots, z_{2t}\} \in V(K_n^{(ct+1)}) \setminus S$ in a cyclic order and the $2t$ subsets $T_1, \ldots, T_{2t}$ consisting of t consecutive vertices in this cyclic order. If for each i, $1 \leq i \leq 2t$ the set T_i is contained in at least $2t$ distinct edges in E_S in color 1, then we would have a t-tight Berge-cycle of length $2t$ in color 1 in F_S, a contradiction. Hence we may assume that for some T_i (say T_1 without loss of generality) apart from at most $(c-u)(n+t)+2t$ exceptional edges all edges in $E_{S\cup T_1}$ are in color $2, 3, \ldots, u$.

Let S' be any set of $|S| + t = (c - u + 1)t$ vertices containing $S \cup T_1 \cup \{z\}$ in the first case and $S \cup T_1$ in the second case. Thus in both cases at least $|E_{S'}| - (c - u + 1)(n + t)$ edges of $E_{S'}$ are colored by at most $u - 1$ colors. But $|E_{S'}| - (c - u + 1)(n + t) = \binom{n-(|S|+t)}{ct+1-(|S|+t)} - (c - (u - 1))(n + t) > 0$, $1 \leq u - 1 = c - \frac{|S'|}{t}$, and $|S'| = (c - u + 1)t$, so by the induction assumption $E_{S'}$ contains a monochromatic Hamiltonian t-tight Berge-cycle, $C_n^{(ct+1,t)}$, contradicting the assumption that E_S contains no monochromatic $C_n^{(ct+1,t)}$. Therefore for any u, $1 \leq u \leq c$, E_S contains a monochromatic $C_n^{(ct+1,t)}$. ■

Now the proof of Theorem 4 is concluded by applying Theorem 7 with $S = \emptyset$. ■

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