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Note

Size of monochromatic components in local edge colorings

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Abstract

An edge coloring of a graph is a local r coloring if the edges incident to any vertex are colored with at most r distinct colors. We determine the size of the largest monochromatic component that must occur in any local r coloring of a complete graph or a complete bipartite graph.

Keywords: Local colorings; Connected components

An easy exercise—in fact a note of Paul Erdős—is that in every 2-coloring of the edges of K_n there is a monochromatic connected subgraph on n vertices. For three colors the analogue problem was solved in [1,9]. The problem was rediscovered in [4]. The generalization of this for r colors is proved by the first author [10]: if the edges of K_n are colored with r colors then there is a monochromatic connected component with at least $n/(r-1)$ vertices. This result also follows from a more general result of Füredi [7]. The result is sharp if $r-1$ is a prime power and $r-1$ divides n. For sharp results when $r-1$ does not divide n and r is small, see [5]. Generalization of the problem for hypergraphs is treated in [8]. Recently some results are obtained for the case when connectivity is replaced by k-connectivity [6,13].

We show how the answer changes if r-coloring is replaced by *local r-coloring*, where the number of colors can be larger than r, but the requirement is that edges incident to any vertex are colored with at most r colors. Ramsey numbers in local r-colorings have been introduced in [11,12].

Let $f(n,r)$ denote the largest m such that in every local r-coloring of the edges of K_n there is a monochromatic connected subgraph with m vertices. This function has been also defined implicitly in [2], where mixed Ramsey numbers introduced. In particular, $RM(\mathcal{T}_n, G)$ was defined as the minimum m such that in any edge coloring of K_m there is either a monochromatic tree on n vertices or a totally multicolored copy of G. The special case when G is a forest on four edges were treated in [3]. Since the requirement of forbidding a multicolored $K_{1,4}$ is equivalent to local 3-colorings, $RM(\mathcal{T}_n, K_{1,4}) \leqslant (7n/3)$ follows from the case $r=3$ of our main result, settling a question raised in [3].

Theorem 1. $f(n,r) \geqslant rn/(r^2-r+1)$ *with equality if a finite plane of order* $r-1$ *exists and* r^2-r+1 *divides* n.

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To show equality in the claimed case, consider the points of a finite plane of order $r-1$ as the vertices of a complete graph and color each pair of vertices by the line going through them. Then replace each vertex i by a k-element set A_i so that the A_i's are pairwise disjoint. The coloring is extended naturally with the proviso that the edges within A_i's are colored with some color among the colors that were incident to vertex i. The result is a locally r-colored K_n where $n = k(r^2 - r + 1)$ and the largest monochromatic connected subgraph has $kr = nr/(r^2 - r + 1)$ vertices.

We give two proofs for Theorem 1. One is based on the following result, perhaps interesting in its own. A *double star* is a tree obtained from two vertex disjoint stars by connecting their centers.

Theorem 2. *Assume that the edges of a complete bipartite graph $G = [A, B]$ are colored so that the edges incident to any vertex of A are colored with at most p colors and the edges incident to any vertex of B are colored with at most q colors. Then there exists a monochromatic connected subgraph H with at least $|A|/q + |B|/p$ vertices. In fact, H can be selected as a double star.*

Corollary 1. *If the edges of a complete bipartite graph G are locally r-colored, there exists a monochromatic connected subgraph (in fact a double star) with at least $|V(G)|/r$ vertices.*

The special case of Corollary 1, when local r-colorings are replaced by usual r-colorings was proved in [10] (without the remark about the double star). A considerably simpler proof (that gives the stronger result about the double star) was given by Liu et al. [13]. We use their method to prove Theorem 2.

Proof of Theorem 2. Let $d_i(v)$ denote the degree of v in color i. For any edge ab of color i, $a \in A, b \in B$, set $c(a, b) = d_i(a) + d_i(b)$. Let $I(v)$ denote the set of colors on the edges incident to $v \in V(G)$. Then, by using the Cauchy–Swartz inequality and the local coloring conditions, we get

$$\sum_{ab \in E(G)} c(a,b) = \sum_{a \in A} \sum_{i \in I(a)} d_i^2(a) + \sum_{b \in B} \sum_{i \in I(b)} d_i^2(b) \geqslant |A| p \left(\frac{\sum_{a\in A} \sum_{i \in I(a)} d_i(a)}{|A|p} \right)^2 + |B| q \left(\frac{\sum_{b\in B} \sum_{i \in I(b)} d_i(b)}{|B|q} \right)^2 = |A||B| \left(\frac{|B|}{p} + \frac{|A|}{q} \right),$$

therefore for some $a \in A$, $b \in B$, $c(a, b) \geqslant |A|/q + |B|/p$. Since the edges incident to a or b in the color of ab span a monochromatic connected subgraph with $c(a, b)$ vertices, Theorem 2 follows. □

Proof of Theorem 1. If any monochromatic, say red component C satisfies $|C| \geqslant rn/(r^2 - r + 1)$, we have nothing to prove. Otherwise apply Theorem 2 for the complete bipartite graph $[A, B] = [V(C), V(G) \setminus V(C)]$. The edges incident to any $v \in A$ are colored with at most $p = r - 1$ colors and the edges incident to any $v \in B$ are colored with at most $q = r$ colors. Thus, by Theorem 2, there is a monochromatic component of size at least

$$|A|/q + |B|/p = \frac{|A|}{r} + \frac{n - |A|}{r-1} = \frac{n}{r-1} - |A| \left(\frac{1}{r-1} - \frac{1}{r} \right) \geqslant$$
$$\geqslant n \left(\frac{1}{r-1} - \frac{r}{r^2 - r + 1} \left(\frac{1}{r(r-1)} \right) \right) = \frac{rn}{r^2 - r + 1}. \qquad \square$$

Our second proof for Theorem 1 applies a result of Füredi [7]. Assume that the edges of K_n are locally r-colored. Consider the hypergraph H whose vertices are the vertices of K_n and whose edges are the vertex sets of the connected monochromatic components. In the dual of H, H^*, every edge has at most r vertices and each pair of edges has a nonempty intersection. Füredi proved [7] that in such hypergraphs the fractional transversal number, $\tau^*(H^*) \leqslant r - 1 + 1/r$. Using well-known elementary facts,

$$\frac{|E(H^*)|}{D(H^*)} \leqslant \nu^*(H^*) = \tau^*(H^*) \leqslant r - 1 + \frac{1}{r},$$

where D is the maximum degree of H^*. Thus we have $r|E(H^*)|/(r^2 - r + 1) \leqslant D(H^*)$. Noting that $|E(H^*)| = n$ and $D(H^*)$ equals to the maximum size of an edge in H, i.e. the maximum size of a connected component in the local r-coloring, the inequality of Theorem 1 follows.

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