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An improved bound for the monochromatic cycle partition number

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Abstract

Improving a result of Erdős, Gyárfás and Pyber for large n we show that for every integer $r \geqslant 2$ there exists a constant $n_0 = n_0(r)$ such that if $n \geqslant n_0$ and the edges of the complete graph K_n are colored with r colors then the vertex set of K_n can be partitioned into at most $100r \log r$ vertex disjoint monochromatic cycles.

Keywords: Edge colorings; Monochromatic partitions; Regularity Lemma

1. Introduction

1.1. Vertex partitions by monochromatic cycles

Assume that K_n is a complete graph on n vertices whose edges are colored with r colors ($r \geqslant 2$). How many monochromatic cycles are needed to partition the vertex set of K_n? Throughout the paper, single vertices and edges are considered to be cycles. Let $p(r)$ denote the minimum

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number of monochromatic cycles needed to partition the vertex set of any r-colored K_n. It is not obvious that $p(r)$ is a well-defined function. That is, it is not obvious that there is always a partition whose cardinality is independent of the order of the complete graph. However, in [5] Erdős, Gyárfás and Pyber proved that there exists a constant c such that $p(r) \leqslant cr^2 \log r$ (throughout this paper log denotes natural logarithm). Furthermore, in [5] (see also [6]) the authors conjectured the following.

Conjecture 1. $p(r) = r$.

The special case $r = 2$ of this conjecture was asked earlier by Lehel and for $n \geqslant n_0$ was proved by Łuczak, Rödl and Szemerédi [13]. Let us also note that the above problem was generalized for complete bipartite graphs (see Haxell [8]) and for vertex partitions by monochromatic connected k-regular subgraphs (see Sárközy and Selkow [16]).

In this paper we give a significant improvement on the above mentioned result of Erdős, Gyárfás and Pyber for large n.

Theorem 1. *For every integer $r \geqslant 2$ there exists a constant $n_0 = n_0(r)$ such that if $n \geqslant n_0$ and the edges of the complete graph K_n are colored with r colors then the vertex set of K_n can be partitioned into at most $100r \log r$ vertex disjoint monochromatic cycles.*

Since Theorem 1 is probably far from best possible, we make no attempt at optimizing the constant 100 in the theorem.

1.2. Sketch of the proof of Theorem 1

A matching in a graph G is called *connected* if its edges are all in the same connected component of G. To prove Theorem 1 we apply the edge-colored version of the Regularity Lemma to an r-colored K_n. Then we introduce the so-called *reduced graph* G^R, the graph whose vertices are associated to the clusters and whose edges are associated to ε-regular pairs. The edges of the reduced graph will be colored with a color that appears on most of the edges between the two clusters. Then we study large monochromatic connected matchings in the reduced graph. That was initiated in [12] and played an important role in our recent paper [7] where we determined the three-color Ramsey numbers of paths for large n.

Generalizing the proof technique in [5], we establish the bound on $p(r)$ in the following steps:

- *Step* 1: We find a sufficiently large monochromatic (say red), dense (more precisely half-dense in a sense explained later), connected matching M in G^R.
- *Step* 2: We remove the vertices of M from G^R and we go back to the original graph (instead of the reduced graph). We greedily remove a number (depending on r) of vertex disjoint monochromatic cycles from the remainder in K_n until the number of leftover vertices is much smaller than the number of vertices associated to M.
- *Step* 3: Using a lemma about cycle covers of r-colored unbalanced complete bipartite graphs (Lemma 6 that may be of independent interest) we combine the leftover vertices with some vertices of the clusters associated with vertices of M.
- *Step* 4: Finally after some adjustments through alternating paths with respect to M, we find a red cycle spanning the remaining vertices of M.

The improvement of Theorem 1 over the result in [5] comes from two factors. First, the matching M plays the role of the triangle cycle in [5], and we are able to find a larger M than the triangle cycle found there. Second, our Lemma 6 in step 3 improves a similar lemma from [5].

The organization of the paper follows this outline. After giving the definitions and tools, we discuss each step one by one.

1.3. Notation and definitions

For basic graph concepts see the monograph of Bollobás [2]. Disjoint union of sets will be sometimes denoted by $+$. $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A + B$, and $E \subset A \times B$. K_n is the complete graph on n vertices, $K(n_1, \dots, n_k)$ is the complete k-partite graph with classes containing $n_1, \dots, n_k$ vertices, P_n (C_n) is the path (cycle) with n vertices. $G(n_1, \dots, n_k)$ is a k-partite graph with classes containing $n_1, \dots, n_k$ vertices. For a graph G and a subset U of its vertices, $G|_U$ is the restriction to U of G. $\Gamma(v)$ is the set of neighbors of $v \in V$. Hence the size of $\Gamma(v)$ is $|\Gamma(v)| = \deg(v) = \deg_G(v)$, the degree of v. $\delta(G)$ stands for the minimum, and $\Delta(G)$ for the maximum degree in G. For a vertex $v \in V$ and set $U \subseteq V - \{v\}$, we write $\deg(v, U)$ for the number of edges from v to U. When A, B are disjoint subsets of $V(G)$, we denote by $e_G(A, B)$ the number of edges of G with one endpoint in A and the other in B. A multi-coloring of a graph G is a coloring where each edge may receive more than one color. For non-empty A and B,

$$d_G(A, B) = \frac{e_G(A, B)}{|A||B|}$$

is the *density* of the graph between A and B.

Definition 1. The bipartite graph $G = (A, B, E)$ is (ε, G)-*regular* if

$$X \subset A,\ Y \subset B,\ |X| > \varepsilon|A|,\ |Y| > \varepsilon|B| \quad \text{imply} \quad \left|d_G(X, Y) - d_G(A, B)\right| < \varepsilon,$$

otherwise it is (ε, G)-*irregular*. Furthermore, (A, B, E) is (ε, δ, G)-*super-regular* if it is (ε, G)-regular and

$$\deg_G(a) > \delta|B| \quad \forall a \in A, \qquad \deg_G(b) > \delta|A| \quad \forall b \in B.$$

1.4. Tools

In the proof an r-color version of the Regularity Lemma and the Blow-up Lemma play a central role.

Lemma 1 *(Regularity Lemma [17]). For every positive ε and positive integer m there are positive integers M and n_0 such that for $n \geqslant n_0$ the following holds. For all graphs $G_1, G_2, \dots, G_r$ with $V(G_1) = V(G_2) = \dots = V(G_r) = V$, $r \geqslant 2$, $|V| = n$, there is a partition of V into $l + 1$ classes (clusters)*

$$V = V_0 + V_1 + V_2 + \dots + V_l$$

such that

- $m \leqslant l \leqslant M$;
- $|V_1| = |V_2| = \cdots = |V_l|$;
- $|V_0| < \varepsilon n$;
- *apart from at most* $\varepsilon\binom{l}{2}$ *exceptional pairs, the pairs* $\{V_i, V_j\}$ *are* (ε, G_s)*-regular for* $s = 1, 2, \ldots, r$.

For an extensive survey on different variants of the Regularity Lemma see [11]. We will also use the following property of (ε, δ, G)-super-regular pairs.

Lemma 2. *For every* $\delta > 0$ *there exist an* $\varepsilon > 0$ *and* m_0 *such that the following holds. Let* G *be a bipartite graph with bipartition* $V(G) = V_1 \cup V_2$ *such that* $|V_1| = |V_2| = m \geqslant m_0$*, and let the pair* (V_1, V_2) *be* (ε, δ, G)*-super-regular. Then for every pair of vertices* $v_1 \in V_1$, $v_2 \in V_2$, G *contains a Hamiltonian path connecting* v_1 *and* v_2.

A lemma somewhat similar to Lemma 2 is used by Łuczak in [12] and by Haxell in [8]. Lemma 2 is a special case of the much stronger Blow-up Lemma (see [9,10]).

We will also use the following simple lemma of Mader [14] (see also [2,3]) on the existence of highly connected subgraphs.

Lemma 3. *Every graph of average degree at least* $4k$ *has a* k*-connected subgraph.*

We will use the following consequence of this lemma. A matching M in a graph G is called *k-half dense* if one can label its edges as $x_1y_1, \ldots, x_{|M|}y_{|M|}$ so that each vertex of $X = \{x_1, \ldots, x_{|M|}\}$ (called the strong end points) is adjacent in G to at least k vertices of $Y = \{y_1, \ldots, y_{|M|}\}$.

Lemma 4. *Every graph* G *of average degree at least* $8k$ *has a connected* k*-half dense matching.*

Proof. Using a well-known remark of Erdős, G has a bipartite subgraph H of average degree at least $4k$. Using Lemma 3, H has a k-connected subgraph $F = [A, B]$, in particular, the minimum degree of F is at least k. Let M be a maximum matching of F, set $A_1 = A \cap M$, $B_1 = B \cap M$. Clearly M is connected. If $A_1 = A$ then M is k-half dense (with $X = B_1$, $Y = A_1$). Otherwise consider the set $A_2 \subseteq A_1$ which can be reached from $A \setminus A_1$ by an alternating path in F with respect to M. Let B_2 denote the other endpoints of the edges of M incident to A_2. Set $A_3 = A_1 \setminus A_2$, $B_3 = B_1 \setminus B_2$. Observe that (from the definition of A_2) no edges of F are in $[A_2, B_3]$, $[A \setminus A_1, B_3]$ and (from M being maximum), no edges of F are in $[A_2, B \setminus B_1]$ either. Therefore M with $X = A_2 \cup B_3$ is a k-half dense matching in $F \subseteq G$. □

It is possible that Lemma 4 can be generalized from half dense matchings to dense matchings where each vertex of Y is also adjacent to at least k vertices of X.

Finally we need the following lemma about dense directed graphs.

Lemma 5. *Let* $\vec{G} = \vec{G}(V, E)$ *be a directed graph with* $|V| = n$ *sufficiently large and minimum out-degree* $d_+(x) \geqslant cn$ *for some constant* $0 < c \leqslant 0.001$. *Then there are subsets* $X, Y \subseteq V$ *such that*

- $|X|, |Y| \geqslant cn/2$;
- *from every $x \in X$ there are at least $c^6 n$ internally vertex disjoint paths of length at most c^{-3} to every $y \in Y$ (denoted by $x \hookrightarrow y$).*

Proof. The relative minimum out-degree (rmo) of a directed graph $\vec{G}$ is the fraction

$$\frac{\min_{x\in V} d_+(x)}{|V|}.$$

We show that if $x \not\hookrightarrow y$ for some high in-degree vertex y then we can always choose a subgraph of $\vec{G}$ with significantly higher rmo. Iterating this argument at most $c^{-1.5}$ times will give the required sets.

Let $\vec{G}_0 = \vec{G}$, $X_0 = V$ and $Y_0 = \{x \in V(\vec{G}_0)\colon d_-(x) \geqslant cn/3\}$. $|X_0| = n$ and

$$|Y_0|n + \big(n - |Y_0|\big)cn/3 \geqslant cn^2$$

follows from the degree condition. Therefore, $|Y_0| \geqslant (2cn/3)/(1-(c/3)) > cn/2$, i.e., X_0 and Y_0 satisfy the size requirement. So if $x \hookrightarrow y$ for every $x \in X_0$ and $y \in Y_0$ then the procedure stops. Else $\exists x \in X_0$ and $y \in Y_0$ s.t. $x \not\hookrightarrow y$.

Given $\vec{G}_{i-1}$, X_{i-1}, Y_{i-1} satisfying the size requirement, the procedure stops if $x \hookrightarrow y$ for every $x \in X_{i-1}$ and $y \in Y_{i-1}$. Otherwise for some x, y, $x \not\hookrightarrow y$, and we define $\vec{G}_i$, X_i, Y_i as follows. Select a maximal system of short (at most c^{-3} long) pairwise internally vertex disjoint paths between x and y. Obtain first $\vec{G}'_i$ by deleting all internal vertices in these short paths. Notice that fewer than $c^{-3}c^6 n = c^3 n$ vertices are removed and there is no short path in $\vec{G}'_i$ from x to y. In $\vec{G}'_i$ find a breadth first search tree T_i with $T_i^0 = \{x\}$, $T_i^1, T_i^2 \ldots,$ where T_i^ℓ is the set of vertices of distance ℓ from x in $\vec{G}'_i$. Observe that there must be some $1 \leqslant j \leqslant (2c^3)^{-1}$ with $|T_i^{j+1}| \leqslant 2c^3 n$. Otherwise

$$\left|\bigcup_{\ell=0}^{(2c^3)^{-1}} T_i^\ell\right| \geqslant \big(2c^3\big)^{-1} 2c^3 n = n,$$

i.e., we can reach from x all vertices of $\vec{G}'_i$, including y and its in-neighbors, via paths of length at most $(2c^3)^{-1} \leqslant c^{-3} - 1$, a contradiction. Let $\vec{G}_i$ be the graph spanned by $\bigcup_{\ell=0}^{j} T_i^\ell$ in $\vec{G}'_i$,

$$X_i = V(\vec{G}_i) \quad \text{and} \quad Y_i = \big\{x \in V(\vec{G}_i)\colon d_-(x) \geqslant cn/3 \text{ in } \vec{G}_i\big\}.$$

Note that $|V(\vec{G}_i)| \leqslant |V(\vec{G}_{i-1})| - cn/3$ since none of the at least $cn/3$ in-neighbors of y lies in $\vec{G}_i$. (This will be used in (2).)

Let d_i (r_i) be the minimum out-degree (rmo) in $\vec{G}_i$. For $i \leqslant c^{-1.5}$

$$d_i \geqslant d_{i-1} - c^6 c^{-3} n - 2c^3 n = d_{i-1} - 3c^3 n \geqslant d_0 - c^{-1.5} 3c^3 n = cn - 3c^{1.5} n. \tag{1}$$

Moreover,

$$|Y_i| \geqslant cn/2,$$

otherwise

$$\big|V(\vec{G}_i)\big|\big(cn - 3c^{1.5}n\big) \leqslant \big|E(\vec{G}_i)\big| < \frac{cn}{2}\big|V(\vec{G}_i)\big| + \left(\big|V(\vec{G}_i)\big| - \frac{cn}{2}\right)\frac{cn}{3},$$

which leads to $|V(\vec{G}_i)|(1 - 18c^{0.5}) \leqslant -cn$. The RS is negative and the first factor of the LS is positive, implying $1 - 18c^{0.5} \leqslant 0$, i.e., $c \geqslant 18^{-2} > 0.001$, a contradiction. So if the procedure terminates in at most $c^{-1.5}$ steps the sizes $|X_i|$ and $|Y_i|$ satisfy the requirements of the lemma. Now we finish by showing that the procedure indeed terminates in at most $c^{-1.5}$ steps. For $i \leqslant c^{-1.5}$

$$r_i = \frac{d_i}{|V(\vec{G}_i)|} \geqslant \frac{d_{i-1} - 3c^3 n}{|V(\vec{G}_{i-1})| - \frac{cn}{3}} = \frac{d_{i-1}}{|V(\vec{G}_{i-1})|} \frac{1 - (3c^3 n)/d_{i-1}}{1 - (cn)/(3|V(\vec{G}_{i-1})|)} \tag{2}$$

$$= r_{i-1} \frac{1 - (3c^3 n)/d_{i-1}}{1 - (cn)/(3|V(\vec{G}_{i-1})|)} \geqslant r_{i-1} \frac{1 - (3c^3 n)/(cn - 3c^{1.5} n)}{1 - (cn)/(3n)} \tag{3}$$

$$\geqslant r_{i-1} \frac{1 - (3c^3 n)/(cn - cn/2)}{1 - (cn)/(3n)} = r_{i-1} \frac{1 - 6c^2}{1 - c/3} \geqslant r_{i-1}\left(1 + \frac{c}{4}\right). \tag{4}$$

Here in (2) we utilized that in each step of our algorithm the out-degrees may decrease by at most $3c^3 n$ and the number of the vertices must decrease by at least $cn/3$. In (3) we used the lower bound (1) for the out-degrees in this range and $|V(\vec{G}_{i-1})| \leqslant n$. The last two inequalities in (4) hold for $c \leqslant 1/36$ and $c \leqslant 1/71$, respectively. Both of these are certainly true if $c < 0.001$. Therefore, by $c < 0.001$

$$1 > r_i \geqslant r_0 \left(1 + \frac{c}{4}\right)^i \geqslant c\left(1 + \frac{c}{4}\right)^i \geqslant c e^{(ic/4.3)},$$

which gives

$$i < 4.3 c^{-1} \log c^{-1} \leqslant c^{-1.5}$$

concluding the proof. □

Since in the proof we showed that $Y \subseteq X$, the following corollary is straightforward.

Corollary 1. *Let $\vec{G} = \vec{G}(V, E)$ be a directed graph with $|V| = n$ and minimum out-degree $d_+(x) = cn$ for some constant $0 < c \leqslant 0.001$. Then there exists $Y \subseteq V$ such that*

- $|Y| \geqslant cn/2$;
- *from every $x \in Y$ there are at least $c^6 n$ internally vertex disjoint paths of length at most c^{-3} to every $y \in Y$.*

2. Proof of Theorem 1

2.1. Step 1

We will assume that n is sufficiently large. We will use the following main parameters:

$$0 < \varepsilon \ll \delta \ll 1, \tag{5}$$

where $a \ll b$ means that a is sufficiently small compared to b. In order to present the results transparently we do not compute the actual dependencies, although it could be done. Although our proof works for $r = 2$, by the result of [13] we assume throughout that $r \geqslant 3$.

Consider an r-edge coloring $(G_1, G_2, \ldots, G_r)$ of K_n. Apply the r-color version of the Regularity Lemma (Lemma 1), with ε as in (5) and get a partition of $V(K_n) = V = \bigcup_{0 \leqslant i \leqslant l} V_i$, where $|V_i| = m$, $1 \leqslant i \leqslant l$. We define the reduced graph G^R: The vertices of G^R are $p_1, \ldots, p_l$, and we have an edge between vertices p_i and p_j if the pair $\{V_i, V_j\}$ is (ε, G_s)-regular for $s = 1, 2, \ldots, r$.

Thus we have a one-to-one correspondence $f: p_i \to V_i$ between the vertices of G^R and the clusters of the partition. Then,

$$\left|E\left(G^R\right)\right| \geqslant (1-\varepsilon)\binom{l}{2},$$

and thus G^R is a $(1-\varepsilon)$-dense graph on l vertices.

Define an edge-coloring $(G_1^R, G_2^R, \ldots, G_r^R)$ of G^R by r colors in the following way. The edge $p_i p_j$ is colored with a color s that contains the most edges from $K(V_i, V_j)$, thus clearly $|E_{G_s}(V_i, V_j)| \geqslant \frac{1}{r}|V_i||V_j|$. Let us take the color class in this coloring that has the most edges. For simplicity assume that this is G_1^R and call this color red. Clearly, we have

$$\left|E\left(G_1^R\right)\right| \geqslant (1-\varepsilon)\frac{1}{r}\binom{l}{2},$$

and thus using (5) the average degree in G_1^R is at least $(1-\varepsilon)(l-1)/r \geqslant l/2r$. Using Lemma 4 we can find a connected $l/16r$-half dense matching M in G_1^R. Say M has size

$$|M| = l_1 \geqslant \frac{l}{16r},$$

and the matching $M = \{e_1, e_2, \ldots, e_{l_1}\}$ is between the two sets of end points U_1 and U_2, where U_1 contains the strong end points, i.e., the points in U_1 have at least $l/16r$ neighbors in U_2. Furthermore, define $f(e_i) = (V_1^i, V_2^i)$ for $1 \leqslant i \leqslant l_1$, where V_1^i is the cluster assigned to the strong end point of e_i, and V_2^i is the cluster assigned to the other end point. Hence we have our large, red, half-dense, connected matching M as desired in step 1.

We need to do some preparations on the matching M. First we will find connecting paths between the edges of the matching M. Since M is a connected matching in G_1^R we can find a connecting path P_i^R in G_1^R from $f^{-1}(V_2^i)$ to $f^{-1}(V_1^{i+1})$ for every $1 \leqslant i \leqslant l_1 - 1$. Note that these paths in G_1^R may not be internally vertex disjoint. From these paths P_i^R in G_1^R we can construct vertex disjoint connecting paths P_i in G_1 connecting a typical vertex v_2^i of V_2^i to a typical vertex v_1^{i+1} of V_1^{i+1}. More precisely we construct P_1 with the following simple greedy strategy. Denote $P_1^R = (p_1, \ldots, p_t)$, $2 \leqslant t \leqslant l$, where according to the definition $f(p_1) = V_2^1$ and $f(p_t) = V_1^2$. Let the first vertex u_1 $(= v_2^1)$ of P_1 be a vertex $u_1 \in V_2^1$ for which $\deg_{G_1}(u_1, f(p_2)) \geqslant (1/r - \varepsilon)m$ and $\deg_{G_1}(u_1, V_1^1) \geqslant (1/r - \varepsilon)m$. By (ε, G_1)-regularity most of the vertices satisfy this in V_2^1. The second vertex u_2 of P_1 is a vertex $u_2 \in (f(p_2) \cap N_{G_1}(u_1))$ for which $\deg_{G_1}(u_2, f(p_3)) \geqslant (1/r - \varepsilon)m$. Again by (ε, G_1)-regularity most vertices satisfy this in $f(p_2) \cap N_{G_1}(u_1)$. The third vertex u_3 of P_1 is a vertex $u_3 \in (f(p_3) \cap N_{G_1}(u_2))$ for which $\deg_{G_1}(u_3, f(p_4)) \geqslant (1/r - \varepsilon)m$. We continue in this fashion, finally the last vertex u_t $(= v_1^2)$ of P_1 is a vertex $u_t \in (f(p_t) \cap N_{G_1}(u_{t-1}))$ for which $\deg_{G_1}(u_t, V_2^2) \geqslant (1/r - \varepsilon)m$.

Then we move on to the next connecting path P_2. Here we follow the same greedy procedure, we pick the next vertex from the next cluster in P_2^R. However, if the cluster has occurred already on the path P_1^R, then we just have to make sure that we pick a vertex that has not been used on P_1.

We continue in this fashion and construct the vertex disjoint connecting paths P_i in G_1, $1 \leqslant i \leqslant l_1 - 1$. These will be parts of the final cycle in G_1. We remove the internal vertices of these paths from G_1. Furthermore, we remove some more vertices from each (V_1^i, V_2^i), $1 \leqslant i \leqslant l_1$, to achieve super-regularity in all of these pairs. From V_1^i we remove all exceptional vertices v_1 for which

$$\deg_{G_1}\bigl(v_1, V_2^i\bigr) < \left(\frac{1}{r} - \varepsilon\right)m,$$

and from V_2^i all exceptional vertices v_2 for which

$$\deg_{G_1}\bigl(v_2, V_1^i\bigr) < \left(\frac{1}{r} - \varepsilon\right)m.$$

(ε, G)-regularity guarantees that at most $\varepsilon|V_j^i|$ vertices are removed from each cluster V_j^i. By doing this we may create some discrepancies in the cardinalities of the clusters of this connected matching. We remove some more vertices from each cluster V_j^i of the matching to assure that now we have the same number of vertices left in each cluster of the matching. For simplicity we still keep the notation $f(e_i) = (V_1^i, V_2^i)$ for the modified clusters. The removed vertices are added to the leftover vertices in $K_n \setminus f(M)$.

Note that at this point we could have a red cycle spanning almost all vertices of $f(M)$. Indeed, by applying Lemma 2 for $1 \leqslant i \leqslant l_1$, we get a path in $G_1|_{f(e_i)}$ connecting v_1^i and v_2^i that contains all of the remaining vertices of $f(e_i)$ (in case of $i = 1$ we just select a Hamiltonian path of $f(e_1)$ starting from v_2^1 and in case of $i = l_1$, we select a Hamiltonian path of $f(e_{l_1})$ starting from $v_1^{l_1}$). However, for technical reasons we postpone the construction of this cycle until the end of step 4, since in step 3 we will use some of the vertices in $f(M)$, and we will have to make some adjustments first in step 4.

2.2. Step 2

Here we will use the easy fact that an r-colored K_n contains a monochromatic cycle of length at least n/r. Indeed, we can use the most frequent color of K_n and apply the Erdős–Gallai extremal theorem for cycles (see [4] or [2]).

We go back from the reduced graph to the original graph and we remove the vertices assigned to the matching M, i.e., $f(M)$. We apply repeatedly the above fact to the r-colored complete graph induced by $K_n \setminus f(M)$. This way we choose t vertex disjoint monochromatic cycles in $K_n \setminus f(M)$. Define the constant $c = 1/350r$. We wish to choose t such that the remaining set B of vertices in $K_n \setminus f(M)$ not covered by these t cycles has cardinality at most $c^{12}n$. Since after t steps at most

$$\bigl(n - |f(M)|\bigr)\left(1 - \frac{1}{r}\right)^t$$

vertices are left uncovered, we have to choose t to satisfy

$$\bigl(n - |f(M)|\bigr)\left(1 - \frac{1}{r}\right)^t \leqslant c^{12}n.$$

This inequality is certainly true if

$$\left(1 - \frac{1}{r}\right)^t \leqslant c^{12},$$

which in turn is true using $1 - x \leqslant e^{-x}$ if

$$e^{-\frac{t}{r}} \leqslant c^{12}.$$

This shows that we can choose $t = 12r\lceil \log 350r\rceil$.

We may assume that the number of remaining vertices in B is even by removing one more vertex (a degenerate cycle) if necessary.

2.3. Step 3

The key to this step is the following lemma about r-colored complete unbalanced bipartite graphs that may be interesting on its own. We will assume $r \geqslant 3$.

Lemma 6. *There exists a constant n_0 such that the following is true. Assume that the edges of the complete bipartite graph $K(A, B)$ are colored with r colors. If $|A| \geqslant n_0$, $|B| \leqslant |A|/r^2$, then B can be covered by at most $(6r\lceil \log r\rceil + 2r)$ vertex disjoint monochromatic cycles.*

We have the connected, red matching M of size l_1 between U_1 and U_2. Define the auxiliary directed graph $\vec{G}$ on the vertex set U_1 as follows. We have the directed edge from V_1^i to V_1^j, $1 \leqslant i, j \leqslant l_1$ if and only if $(V_1^i, V_2^j) \in G_1^R$. The fact that M is $l/16r$-half dense implies that in $\vec{G}$ for the minimum out-degree we have

$$\min_{x \in U_1} d_+(x) \geqslant \frac{l}{16r} \geqslant \frac{|U_1|}{16r} \quad \left(\geqslant \frac{|U_1|}{350r}\right).$$

Thus applying Lemma 5 for $\vec{G}$ with $c = \frac{1}{350r}$ (< 0.001), there are subsets $X_1, Y_1 \subset U_1$ such that

- $|X_1|, |Y_1| \geqslant c|U_1|/2$;
- from every $x \in X_1$ there are at least $c^6|U_1|$ internally vertex disjoint paths of length at most c^{-3} to every $y \in Y_1$ $(x \hookrightarrow y)$.

Let X_2, Y_2 denote the set of the other endpoints of the edges of M incident to X_1, Y_1, respectively. Note that a path in $\vec{G}$ corresponds to an alternating path with respect to M in G_1^R.

In each cluster $V_1^i \in Y_1$ let us consider an arbitrary subset of $c^8|V_1^i|$ vertices. Let us denote by A_1 the union of all of these subsets. Similarly we denote by A_2 the union of arbitrary subsets of $V_2^j \in X_2$ of size $c^8|V_2^j|$. Then we have

$$|A_1|, |A_2| \geqslant c^8\left|f(Y_1)\right| \geqslant c^8\frac{c}{2}\left|f(U_1)\right| \geqslant c^8\frac{c}{2}\frac{n}{16r} \geqslant c^{10}n.$$

Let us divide the remaining vertices in B (B was defined in step 2) into two equal sets B_1 and B_2. Thus we have $|B_1|, |B_2| \leqslant |B| \leqslant c^{12}n$. We apply Lemma 6 in $K(A_1, B_1)$ and in $K(A_2, B_2)$. The conditions of the lemma are satisfied by the above since $|B_i| \leqslant |A_i|/r^2$ for $i = 1, 2$. Let us remove the at most $6r\lceil \log r\rceil + 2r \leqslant 8r\lceil \log r\rceil$ vertex disjoint monochromatic cycles covering B_1 in $K(A_1, B_1)$ and the at most $8r\lceil \log r\rceil$ cycles covering B_2 in $K(A_2, B_2)$. By doing this we may create discrepancies in the number of remaining vertices in the two clusters of a matching edge. In the next step we have to eliminate these discrepancies with the use of the many alternating paths.

2.4. Step 4

By removing the vertex disjoint monochromatic cycles covering B_1 in $K(A_1, B_1)$ we have created a "surplus" of $|B_1|$ vertices in the clusters of Y_2 compared to the remaining number of vertices in the corresponding clusters of Y_1. Similarly by removing the cycles covering B_2 in $K(A_2, B_2)$ we have created a "deficit" of $|B_2|$ $(= |B_1|)$ vertices in the clusters of X_2 compared to the number of vertices in the corresponding clusters of X_1. The natural idea is to "move" the surplus from Y_2 through an alternating path to cover the deficit in X_2.

Take an arbitrary cluster $V_2^i \in Y_2$ that has a surplus of $0 < s$ $(\leqslant c^8|V_2^i|)$ vertices and an arbitrary cluster $V_2^j \in X_2$ that has a deficit of $0 < d$ $(\leqslant c^8|V_2^j|)$ vertices (there must be one such a cluster since the total surplus is equal to the total deficit). Assume $s \leqslant d$ and we will move a surplus of size s from V_2^i to V_2^j. Otherwise in case $s > d$ we will only move a surplus of size d. By the construction there is an alternating path

$$V_2^j,\ V_1^j,\ V_2^{j_1},\ V_1^{j_1},\ \ldots,\ V_2^{j_k},\ V_1^{j_k},\ V_2^i$$

such that $k < c^{-3}$. We extend the red (G_1) connecting path P_{j-1} (defined in step 1) first by a path of length 2 in the bipartite graph $G_1|_{V_1^j \times V_2^j}$ in such a way that the new endpoint has many neighbors in $V_2^{j_1}$ (ε-regularity makes this possible), and then by a path of length $2s$ in the bipartite graph $G_1|_{V_1^j \times V_2^{j_1}}$. Similarly we extend by a path of length $2s+2$ the red connecting paths P_{j_1-1}, P_{j_2-1}, etc. Finally we extend the red connecting path P_{j_k-1} first by a path of length 2 in the bipartite graph $G_1|_{V_1^{j_k} \times V_2^{j_k}}$, and then by a path of length $2s$ in the bipartite graph $G_1|_{V_1^{j_k} \times V_2^i}$. The overall effect of these extensions is that we moved the surplus of size s from V_2^i to V_2^j without changing any of the other relative sizes in the edges of the matching. This way we came closer to eliminating the discrepancies, and by iterating this procedure we can totally eliminate them.

However, we have to pay attention again that during this process we never use up to many vertices from any given cluster. It is not hard to see from the construction that we can guarantee that during the whole process with these extensions we use up at most $5c^2$-fraction of any given cluster. Indeed, the total number of vertices along these extensions is at most

$$2c^{-3}c^{12}n = 2c^9 n. \tag{6}$$

We declare an alternating path forbidden if there is a cluster along the path from which we used up at least a $4c^2$-fraction already with these extensions. Then by (6) the total number of vertex disjoint forbidden alternating paths during the whole process is at most $\frac{c^7}{2}l$, and thus by Lemma 5 we have plenty of non-forbidden alternating paths to choose from between any V_2^j and V_2^i.

Hence after the whole process the remaining vertices in any matching edge $f(e_i) = (V_1^i, V_2^i)$ still form a super-regular balanced pair with somewhat weaker parameters (say $(2\varepsilon, 1/2r)$-super-regular). Then as we mentioned at the end of step 1 we can close the red cycle to span all the remaining vertices of $f(M)$.

Thus the total number of vertex disjoint monochromatic cycles we used to partition the vertex set of K_n is at most

$$12r\lceil\log(350r)\rceil + 8r\lceil\log r\rceil + 8r\lceil\log r\rceil + 2 \leqslant 100r\lceil\log r\rceil,$$

finishing the proof of Theorem 1.

2.5. Cycle cover lemmas for unbalanced bipartite graphs; proof of Lemma 6

Lemma 6 clearly follows from the following two lemmas.

Lemma 7. *There exists a constant n_0 such that the following is true. Assume that the edges of the complete bipartite graph $K(A,B)$ are colored with r colors. If $|A| \geqslant n_0$, $|B| \leqslant |A|/r^2$, then apart from a set $B' \subset B$ of at most $|A|/2(8r)^{8(r+1)}$ vertices B can be covered by at most $6r\lceil \log r \rceil$ vertex disjoint monochromatic cycles.*

Lemma 8. *There exists a constant n_0 such that the following is true. Assume that the edges of the complete bipartite graph $K(A,B)$ are colored with r colors. If $|A| \geqslant n_0$, $|B| \leqslant |A|/(8r)^{8(r+1)}$, then B can be covered by at most $2r$ vertex disjoint monochromatic cycles.*

We will also use the following simple lemma about the case when B is significantly smaller than A.

Lemma 9. *Assume that the edges of the complete bipartite graph $K(A,B)$ are colored with r colors. If $(|B|-1)r^{|B|} < |A|$, then B can be covered by at most r vertex disjoint monochromatic cycles.*

Let us start with the simple proof of this last lemma.

Proof of Lemma 9. Denote the vertices of B by $\{b_1, b_2, \ldots, b_{|B|}\}$. To each vertex $v \in A$ we assign a vector $(v_1, v_2, \ldots, v_{|B|})$ of colors, where v_i is the color of the edge (v, b_i). The total number of distinct color vectors possible is $r^{|B|}$. Since we have $|A| > (|B|-1)r^{|B|}$ vectors, by the pigeon-hole principle we must have a vector that is repeated at least

$$\frac{|A|}{r^{|B|}} \geqslant |B|$$

times. In other words, there are at least $|B|$ vertices in A for which the colorings of the edges going to $\{b_1, b_2, \ldots, b_{|B|}\}$ are exactly the same. It is easy to see that this gives a covering of B by at most r vertex disjoint monochromatic cycles (in fact complete bipartite graphs can be used instead cycles). □

Proof of Lemma 7. This lemma in turn will use the Regularity Lemma as well. We proceed similarly as in step 1. Consider an r-edge coloring $(G_1, G_2, \ldots, G_r)$ of $K(A,B)$. We know that $|A|$ is sufficiently large and $|B| > |A|/(8r)^{8(r+1)}$, since otherwise we are done by Lemma 8. Then we can apply the bipartite r-color version of the Regularity Lemma (see, e.g., [15]), with ε as in (5). By standard arguments we may assume that for each cluster that is not V_0, all vertices of the cluster belong to the same partite class. Thus we get a partition $A = V_A^0 + V_A^1 + \cdots + V_A^{l_A}$, $B = V_B^0 + V_B^1 + \cdots + V_B^{l_B}$, where $|V_A^{j_1}| = |V_B^{j_2}| = m$, $1 \leqslant j_1 \leqslant l_A$, $1 \leqslant j_2 \leqslant l_B$ and $|V_A^0| \leqslant \varepsilon|A|$, $|V_B^0| \leqslant \varepsilon|B|$. We define again the reduced graph G^R: The vertices of G^R are $A^R = \{p_A^{j_1} \mid 1 \leqslant j_1 \leqslant l_A\}$ and $B^R = \{p_B^{j_2} \mid 1 \leqslant j_2 \leqslant l_B\}$, and we have an edge between vertices $p_A^{j_1}$ and $p_B^{j_2}$, if the pair $\{V_A^{j_1}, V_B^{j_2}\}$ is (ε, G_s)-regular for $s = 1, 2, \ldots, r$. Thus we have a one-to-one correspondence $f : \{p_A^j, p_B^j\} \to \{V_A^j, V_B^j\}$ between the vertices of G^R and the non-exceptional clusters of the partition. Then $G^R = (A^R, B^R)$ is a $(1-\varepsilon)$-dense bipartite graph. Define an r-edge coloring $(G_1^R, G_2^R, \ldots, G_r^R)$ of G^R in the following way. The edge between the clusters $V_A^{j_1}$ and $V_B^{j_2}$ is colored with a color s that contains the most edges from $K(V_A^{j_1}, V_B^{j_2})$.

For each $p_B^j \in B^R$, $1 \leqslant j \leqslant l_B$, and for each color class that contains at most $|l_B|$ edges incident to p_B^j, we delete the edges in this color incident to p_B^j. Thus the number of remaining edges is at least

$$(1-\varepsilon)l_A l_B - r l_B^2 = l_A l_B\left((1-\varepsilon) - r\frac{l_B}{l_A}\right). \tag{7}$$

Then (7) implies that there must be a vertex $p_A^j \in A^R$ that has at least

$$l_B\left((1-\varepsilon) - r\frac{l_B}{l_A}\right) \tag{8}$$

neighbors in B^R. From the definition of edge deletion, the neighbors of p_A^j in each color can be covered by at most r vertex disjoint monochromatic matchings in G^R. Furthermore, these matchings will be connected through p_A^j. Similarly as in step 1, going back to the original graph, from these monochromatic connected matchings we can construct monochromatic cycles that cover most of the clusters belonging to these connected matchings. Thus using (8) we have at most r vertex disjoint monochromatic cycles that cover B apart from at most

$$3\varepsilon|B| + \frac{r}{1-\varepsilon}\frac{|B|^2}{|A|} \tag{9}$$

vertices. Remove the above at most r cycles from (A, B) and denote the resulting sets by (A_1, B_1) (where $A_0 = A$, $B_0 = B$). Then using (9), (5) and

$$|A_1| \geqslant |A| - |B| \geqslant \left(1 - \frac{1}{r^2}\right)|A|$$

we get

$$\begin{aligned}\frac{|B_1|}{|A_1|} &\leqslant 3\varepsilon\frac{|B|}{|A_1|} + \frac{r}{1-\varepsilon}\frac{|B|^2}{|A||A_1|} \leqslant \frac{3\varepsilon r^2}{r^2-1}\frac{|B|}{|A|} + \frac{r^3}{(1-\varepsilon)(r^2-1)}\left(\frac{|B|}{|A|}\right)^2 \\ &\leqslant \frac{3\varepsilon}{r^2-1} + \frac{1}{r(1-\varepsilon)(r^2-1)} \leqslant \frac{2}{r^3}.\end{aligned}$$

We apply repeatedly the above procedure in (A_1, B_1). After k iterations we have

$$\frac{|B_k|}{|A_k|} \leqslant \frac{1}{4}\left(\frac{2}{r}\right)^{2^k+1}.$$

This implies that after $6\lceil\log r\rceil$ iterations (and so with $6r\lceil\log r\rceil$ cycles) we covered B apart from at most

$$\frac{1}{4}\left(\frac{2}{r}\right)^{r^6+1}|A| \leqslant \frac{|A|}{2(8r)^{8(r+1)}}$$

vertices (using $r \geqslant 3$ and some calculation), and thus finishing Lemma 7. □

Proof of Lemma 8. We proceed similarly as in the proof of Lemma 7. Consider an r-edge coloring $(G_1, G_2, \ldots, G_r)$ of $K(A, B)$. A is sufficiently large and we may assume that B is sufficiently large as well, since otherwise we are done by Lemma 9. We may assume that

$$|A| = (8r)^{8(r+1)}|B| \tag{10}$$

by keeping a subset of A of this size and deleting the rest. We apply the bipartite r-color version of the Regularity Lemma and similarly as in the proof of Lemma 7 we get the reduced graph $G^R = (A^R, B^R)$ that is an $(1-\varepsilon)$-dense bipartite graph. However, here we will use a multi-coloring in G^R. Define an r-edge multi-coloring $(G_1^R, G_2^R, \dots, G_r^R)$ of G^R in the following way. The edge between the clusters $V_A^{j_1}$ and $V_B^{j_2}$ has color s for all s such that

$$\left|E_{G_s}\left(V_A^{j_1}, V_B^{j_2}\right)\right| \geqslant \delta \left|V_A^{j_1}\right|\left|V_B^{j_2}\right|.$$

Claim 1. *There exists a color (say G_1, called red) such that $G_1^R = (A^R, B^R)$ contains a connected (A''^R, B''^R) satisfying the following:*

$$\left|A''^R\right| \geqslant \frac{1}{4(4r)^4} l_A \quad \text{and} \quad \left|B''^R\right| \geqslant \frac{1}{2(4r)^2} l_B, \tag{11}$$

$$\deg_{G_1^R}\left(p_B^j, A^R\right) \geqslant \frac{1}{4r} l_A \quad \forall p_B^j \in B''^R, \tag{12}$$

$$\deg_{G_1^R}\left(p_A^j, B''^R\right) \geqslant \frac{1}{2(4r)^2} l_B \quad \forall p_A^j \in A''^R. \tag{13}$$

Proof. There must be a color (say G_1^R, called red) for which

$$\left|E\left(G_1^R\right)\right| \geqslant \frac{1-\varepsilon}{r} l_A l_B \geqslant \frac{1}{2r} l_A l_B. \tag{14}$$

Then there must be a subset $B'^R \subset B^R$ such that

$$\left|B'^R\right| \geqslant \frac{1}{4r} l_B$$

and for every $p_B^j \in B'^R$ we have

$$\deg_{G_1^R}\left(p_B^j\right) \geqslant \frac{1}{4r} l_A.$$

Indeed, otherwise we get

$$\left|E\left(G_1^R\right)\right| = \sum_{p_B^j \in B^R} \deg_{G_1^R}\left(p_B^j\right) < \frac{1}{4r} l_A l_B + \frac{1}{4r} l_A l_B = \frac{1}{2r} l_A l_B,$$

a contradiction with (14). Thus

$$\left|E\left(G_1^R|_{(A^R, B'^R)}\right)\right| \geqslant \frac{1}{(4r)^2} l_B l_A.$$

Similarly, there must be a subset $A'^R \subset A^R$ such that

$$\left|A'^R\right| \geqslant \frac{1}{2(4r)^2} l_A$$

and for every $p_A^j \in A'^R$ we have

$$\deg_{G_1^R}\left(p_A^j, B'^R\right) \geqslant \frac{1}{2(4r)^2} l_B.$$

Now it is not hard to see that we can pick a connected component (A''^R, B''^R) of (A'^R, B'^R) which satisfies the requirements of the claim. Indeed, define the auxiliary graph G_A on the vertex set A'^R as follows. For $x, y \in A'^R$, xy is an edge of G_A if and only if $N_{G_1^R}(x, B'^R) \cap N_{G_1^R}(y, B'^R) \neq \emptyset$.

Fact 1. *The maximum number of pairwise non-adjacent vertices in G_A is at most* $2(4r)^2$.

Then clearly there is a connected component A''^R in G_A of size at least

$$\frac{1}{2(4r)^2}\left|A'^R\right| \geqslant \frac{1}{4(4r)^4} l_A.$$

With the choice $B''^R = N_{G_1^R}(A''^R) \cap B'^R$ we get the connected component (A''^R, B''^R) of G_1^R proving the claim. □

We modify B''^R in the following way. We add any vertex $p_B^j \in (B^R \setminus B''^R)$ to B''^R for which we have

$$\deg_{G_1^R}\left(p_B^j, A''^R\right) \geqslant \frac{2}{(8r)^{8(r+1)}} l_A \quad (\geqslant l_B). \tag{15}$$

For simplicity we keep the notation B''^R for the resulting set. Thus now we may assume that for any $p_B^j \in (B^R \setminus B''^R)$ inequality (15) does not hold.

Then using (12) and (15) by Hall's theorem we can find a monochromatic (red) connected matching M covering the vertices B''^R (note that the other endpoints of the matching edges may not be in A''^R for the original vertices of B''^R).

Denote the found $M = \{e_1, e_2, \ldots, e_{l_1}\}$ and $f(e_i) = (V_A^i, V_B^i)$ for $1 \leqslant i \leqslant l_1$ (where $l_1 = |B''^R|$). Similarly as in step 1 we make the pairs of clusters belonging to the edges in M super-regular (in red). The exceptional vertices removed from the clusters in B are added to V_B^0. Again, similarly as in step 1 we find the connecting red paths between the super-regular pairs belonging to edges of M and we make the partite sets equal inside one super-regular pair. However, we postpone the closing of the red cycle inside each pair of clusters belonging to edges of M. First we need some technical steps. We go back to the original graph and we consider the set of remaining vertices in B:

$$B_1 = V_B^0 + f\left(B^R \setminus B''^R\right).$$

Consider those vertices $v \in B_1$ for which

$$\deg_{G_1}\left(v, f\left(A''^R\right)\right) \geqslant \frac{4}{(8r)^{5(r+1)}}|A| \quad \left(\geqslant |B|\right). \tag{16}$$

These vertices are removed from B_1 and they will be inserted into the red cycle. (For simplicity we will keep the notation B_1 for the remaining vertices.) For this purpose first we need an estimate on the number of vertices satisfying (16). We have $|V_B^0| \leqslant 2\varepsilon|B|$. Let us consider a $p_B^j \in B^R \setminus B''^R$. Using (5), the definition of the coloring in G^R and the fact that for p_B^j (15) does not hold, the number of red edges between $f(p_B^j)$ and $f(A''^R)$ is at most

$$\frac{2}{(8r)^{8(r+1)}} l_A m^2 + \delta l_A m^2 \leqslant \left(\frac{2}{(8r)^{8(r+1)}} + \delta\right)|A|m \leqslant \frac{4}{(8r)^{8(r+1)}}|A|m.$$

This clearly implies that we can have at most $\frac{1}{(8r)^{3(r+1)}}m$ vertices in $f(p_B^j)$ satisfying (16). Thus altogether the number of vertices satisfying (16) is at most

$$\left|V_B^0\right| + \frac{1}{(8r)^{3(r+1)}}\left|f\left(B^R \setminus B''^R\right)\right| \leqslant 2\varepsilon|B| + \frac{1}{(8r)^{3(r+1)}}|B| \leqslant \frac{2}{(8r)^{3(r+1)}}|B|. \tag{17}$$

To handle the vertices satisfying (16) we are going to extend some of the red connecting paths P_i connecting the edges of M so now they are going to include these vertices. Take the first vertex v satisfying (16). Then clearly there is a cluster $p_A^j \in A''^R$ that is not covered by M for which

$$\deg_{G_1}\big(v, f\big(p_A^j\big)\big) \geqslant \delta m.$$

Take an arbitrary neighbor of p_A^j in B''^R (there must be many by (13)). Then this neighbor is covered by the matching M, say by the edge e_i, $1 \leqslant i \leqslant l_1$, where $f(e_i) = (V_A^i, V_B^i)$. Consider the red connecting path P_{i-1} between $f(e_{i-1})$ and $f(e_i)$ ending at the vertex $v_1^i \in V_A^i$. Extend this path by a red path of length 6 such that the third vertex is v and the other vertices come from the following clusters (in this order):

$$V_B^i,\ f\big(p_A^j\big),\ v,\ f\big(p_A^j\big),\ V_B^i,\ V_A^i.$$

For simplicity we still denote the new endpoint (a typical vertex of V_A^i) by v_1^i.

We repeat the same procedure for all the other vertices satisfying (16). However, we have to pay attention to several technical details. First, of course in repeating this procedure we always consider the remaining free vertices in each cluster; the internal vertices of the connecting paths are always removed. Second, we make sure that we never use up too many vertices from any cluster. It is not hard to see (using (13), (16) and (17)) that we can guarantee that we use up at most half of the vertices from every cluster. Finally, since we are removing vertices from a pair (V_A^i, V_B^i), we might violate the super-regularity. Note that we never violate the ε-regularity. Therefore, we do the following. After using up, say, $\lfloor \delta^2 m \rfloor$ vertices from a pair (V_A^i, V_B^i), we update the pair as follows. In the pair (V_A^i, V_B^i) we remove all vertices u from V_A^i (and similarly from V_B^i) for which $\deg(u, V_B^i) < (\delta - \varepsilon)|V_B^i|$ (again, we consider only the remaining vertices). We add the at most εm vertices removed from V_B^i to V_B^0, check whether they satisfy (16) and if they do, we process them with the above procedure.

This way we can handle all the vertices satisfying (16). Now by applying Lemma 2 we can close the red cycle inside each super-regular edge of M such that it covers all the remaining vertices in V_B^i. Indeed, by the above procedure the number of remaining vertices in V_B^i is less than the number of remaining vertices in V_A^i, since in each extending path of length 6, we remove 2 vertices from V_B^i and only one from V_A^i.

Remove this red cycle. Denote the resulting sets by B_1 in B and by A_1 in $f(A''^R)$. Put $A_0 = A$ and $B_0 = B$. By (10), (11) and the fact that the relative proportions in the original graph are almost the same as in the reduced graph we certainly have

$$|A_1| \geqslant \frac{1}{(8r)^4}|A_0|. \tag{18}$$

We will apply repeatedly the above procedure in (A_1, B_1). However, we consider only the $(r-1)$-edge multi-coloring $(G_2, \dots, G_r)$ in $K(A_1, B_1)$, the edges in G_1 are deleted. Notice that $|A_1|$ is still sufficiently large. We have three cases depending on the size B_1.

Case 1. $(|B_1| - 1)r^{|B_1|} < |A_1|$.

In this case we are done by Lemma 9 since we have a covering of B by $r + 1$ ($\leqslant 2r$) vertex disjoint monochromatic cycles. Thus we may assume that this case does not hold.

Case 2. $(8r)^{8(r+1)}|B_1| \leqslant |A_1| \leqslant (|B_1|-1)r^{|B_1|}$.

In this case we may run into the problem that the removed cycle may contain almost all vertices of B, i.e., $|B_1| = o(|A_1|)$. In this case the reduced graph might become empty. To avoid this we keep a subset of A_1 of size $(8r)^{8(r+1)}|B_1|$ (denoted again by A_1) and we delete the rest. By the fact that (16) does not hold we know that before this deletion all vertices in B_1 have small degrees in the color removed (red). But then it may happen that the relative degree (the fraction of the degree and the "new" $|A_1|$) of some vertices in the trimmed B_1 in red will not be small any more, i.e., similarly to (16)

$$\deg_{G_1}(v, A_1) \geqslant \frac{4}{(8r)^{5(r+1)}}|A_1|. \tag{19}$$

To avoid this we choose a *random subset* of A_1 of this size (denoted again by A_1 for simplicity). Then the relative degrees of the vertices of B_1 will be roughly the same as before the deletion of the superfluous vertices. To make this precise we claim the following.

Claim 2. *Let $V_n = \{v_1, \ldots, v_n\}$ with n sufficiently large, $\mathcal{F} = \{S_1, \ldots S_m\}$ with $S_i \subseteq V_n$, $|S_i| \leqslant cn$ for some constant $0 < c \leqslant 1$. Then for arbitrary $k > \frac{3}{c}\log m$ there exists a $T \subseteq V_n$ such that*

- $|T| = k$;
- $|S_i \cap T| \leqslant 2ck \ \forall i$.

Proof. Clearly, by adding arbitrary elements to S_i (and then deleting from the intersection) we may also assume that $|S_i| = cn \ \forall i$. We will use a Chernoff bound, see, e.g. [1]:

$$\Pr\bigl(\mathrm{Bin}(n, p) \geqslant (1+\lambda)np\bigr) \leqslant e^{-\lambda^2 np/3} \quad \text{for } 0 \leqslant \lambda \leqslant 1. \tag{20}$$

Choose uniformly $T \subseteq V_n$ of size k. Let $1_{x\in T}$ be the indicator variable, and

$$|S_i \cap T| = \sum_{x\in S_i} 1_{x\in T}.$$

Now $\Pr(x \in T) = k/n$ and so $|S_i \cap T|$ is dominated by $\mathrm{Bin}(cn, k/n)$ for every i. Using (20) we see that

$$\Pr\bigl(|S_i \cap T| \geqslant 2ck\bigr) \leqslant e^{-ck/3}.$$

Therefore,

$$\Pr\left(\bigvee_{i=1}^{m}\bigl(|S_i \cap T| \geqslant 2ck\bigr)\right) \leqslant \sum_{i=1}^{m}\Pr\bigl(|S_i \cap T| \geqslant 2ck\bigr) \leqslant me^{-ck/3}. \tag{21}$$

If the right-hand side of (21) is less than one then the claimed T must exist, i.e.,

$$e^{-ck/3} < (m)^{-1}, \quad \text{i.e.,} \tag{22}$$

$$k > \frac{3}{c}\log m, \tag{23}$$

which is a requirement of the claim. □

We will apply Claim 2 with the following choices. Let $n = |A_1|$, $V_n = A_1$, $m = |B_1|$, $S_i = N_{G_1}(v_i, A_1)$ for $v_i \in B_1$. Then from (18) and the fact that (16) does not hold it follows that we can select

$$c = \frac{\deg_{G_1}(v_i, A_1)}{|A_1|} < \frac{4}{(8r)^{5(r+1)}} \frac{|A|}{|A_1|} \leqslant \frac{4(8r)^4}{(8r)^{5(r+1)}}. \tag{24}$$

Clearly all the conditions of the claim are satisfied so we can select the desired subset of A_1 of size $(8r)^{8(r+1)}|B_1|$.

Case 3. $|A_1| < (8r)^{8(r+1)}|B_1|$.

In this case we continue with A_1 with no modifications.

Now we are ready to repeat the above procedure in (A_1, B_1). Note that in Case 3 technically we have a somewhat weaker condition for $|A_1|$ in terms of $|B_1|$ compared to the original $|A_0| = (8r)^{8(r+1)}|B_0|$, but that does not create any difficulties, the procedure still goes through.

We will treat Cases 2 and 3 simultaneously. We apply the bipartite $(r-1)$-color version of the Regularity Lemma for the $(r-1)$-colored bipartite graph between A_1 and B_1. Using the fact that in B_1 (16) does not hold and Claim 2 in Case 2, in both Cases 2 and 3 in (7) we still have the $\frac{1}{2r} l_{A_1} l_{B_1}$ lower bound for a color, say G_2^R. In Case 3, the above procedure goes through exactly the same way for (A_1, B_1). Note that in Case 3 in (15) we keep the original

$$\frac{2}{(8r)^{8(r+1)}} l_A$$

lower bound (and we do not use l_{A_1} instead of l_A), and similarly in (16) we keep the

$$\frac{4}{(8r)^{5(r+1)}} |A|$$

lower bound (and we do not use $|A_1|$ instead of $|A|$). However, in Case 2 we replace l_A with l_{A_1} in (15) and $|A|$ with $|A_1|$ in (16). Thus in both cases similarly to (18) we have

$$|A_2| \geqslant \frac{1}{(8r)^4} |A_1|,$$

and furthermore if we had Case 3 for (A_1, B_1), then using (18) we have

$$|A_2| \geqslant \frac{1}{(8r)^4} |A_1| \geqslant \frac{1}{(8r)^8} |A_0|.$$

However, note that if we had Case 2 for (A_1, B_1), then this last inequality might not hold as the "new" $|A_1|$ might be significantly smaller than $\frac{1}{(8r)^4}|A_0|$.

In general let us consider the situation after k iterations in (A_k, B_k). Assume that the last time Case 2 occurred was at k' ($\leqslant k$). If Case 2 never occurred we put $k' = 0$. The above procedure goes through exactly the same way for (A_k, B_k) but we replace l_A with $l_{A_{k'}}$ in (15) and $|A|$ with $|A_{k'}|$ in (16).

If the procedure terminates after k ($\leqslant r$) iterations with no more vertices remaining in B, then we have a cover of B with at most $2r$ vertex disjoint monochromatic cycles, as desired. Assuming that the procedure does not terminate after r iterations, so $B_r \neq \emptyset$, we will get a contradiction. Indeed, let us examine the maximum degree to the set A_r in any color for each vertex $v \in B_r$. For G_1 since (16) does not hold we have

$$\deg_{G_1}(v, A_0) < \frac{4}{(8r)^{5(r+1)}} |A_0|.$$

Then as we saw in (24) in case we have Case 3 for (A_1, B_1) we have

$$\deg_{G_1}(v, A_1) < \frac{4(8r)^4}{(8r)^{5(r+1)}}|A_1|,$$

and in case we have Case 2 for (A_1, B_1) using Claim 2 we have to multiply by an extra factor of 2 to get

$$\deg_{G_1}(v, A_1) < \frac{8(8r)^4}{(8r)^{5(r+1)}}|A_1|.$$

We continue in this fashion, in each iteration we have to multiply the coefficient of $|A_i|$ by a factor of $(8r)^4$ and in addition if it was an iteration where we applied Case 2, then we have to multiply by another factor of 2. Thus for each vertex $v \in B_r$ we have

$$\deg_{G_1}(v, A_r) < \frac{4(2)^r(8r)^{4r}}{(8r)^{5(r+1)}}|A_r| < \frac{|A_r|}{r}.$$

In this upper bound we assumed the worst possible case when we have a Case 2 application in each iteration and that is why we get the extra factor of 2^r. Note also that we have this upper bound for the other colors as well, and thus for each vertex $v \in B_r$ and color $1 \leqslant i \leqslant r$ we have

$$\deg_{G_i}(v, A_r) < \frac{|A_r|}{r},$$

a contradiction, since in at least one of the colors we must have at least $|A_r|/r$ edges from v to A_r. This finishes the proof of Lemma 8. □

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