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Discrete Mathematics 297 (2005) 190–195
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

On a Turán-type hypergraph problem of Brown, Erdős and T. Sós

Gábor N. Sárközy, Stanley Selkow

Computer Science Department, Worcester Polytechnic Institute, Worcester, MA 01609, USA

Received 11 February 2003; received in revised form 25 March 2005; accepted 7 April 2005
Available online 23 June 2005

Abstract

We let $G^{(r)}(n, m)$ denote the set of r-uniform hypergraphs with n vertices and m edges, and $f^{(r)}(n, p, s)$ is the smallest m such that every member of $G^{(r)}(n, m)$ contains a member of $G^{(r)}(p, s)$. In this paper we are interested in the growth of $f^{(r)}(n, p, s)$ for fixed values r, p and s. Brown, Erdős and Sós [Some external problems on r-graphs, in: New Directions in the Theory of Graphs, Proceedings of the Third Annual Arbor Conference on Graph Theory, Academic Press, New York, 1973, pp. 55–63] proved that for $r > k \geqslant 2$ and $s \geqslant 3$ we have $f^{(r)}(n, s(r-k)+k, s) = \Theta(n^k)$. This suggests the difficult question whether $f^{(r)}(n, s(r-k)+k+1, s) = o(n^k)$. This was first established for $r = s = 3$ and $k = 2$ by Ruzsa and Szemerédi [Trible systems with no six points carrying three triangles, in: Combinatorics (Keszthely, 1976), Colloquia Mathematics Societies Janos Bolyai, vols. II, 18, 1976, pp. 939–945]. Then for $s = 3$ and $k = 2$ Erdős et al. [Graphs Combin. 2 (1986) 113–121] extended this result for any r, and they conjectured that it also holds for $k = 2$ and any s. In this note we show that it holds with $\lfloor \log_2 s \rfloor$ in place of 1, i.e.,

$$f^{(r)}(n, s(r-k)+k+\lfloor \log_2 s \rfloor, s) = o(n^k) \quad \text{for all } r > k \geqslant 2 \text{ and } s \geqslant 3.$$

In addition we show that the conjecture holds for $r > k \geqslant 3$ and $s = 4$, i.e.,

$$f^{(r)}(n, 4(r-k)+k+1, 4) = o(n^k).$$

Keywords: Hypergraph Turan density

E-mail addresses: gsarkozy@cs.wpi.edu (G.N. Sárközy), sms@cs.wpi.edu (S. Selkow).

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doi:10.1016/j.disc.2005.05.002

1. Introduction

1.1. Notation and definitions

For basic graph and hypergraph concepts see the monograph of Bollobás [1].

A hypergraph $\mathscr{F}$ is called *r-uniform* if $|F| = r$ for every edge $F \in \mathscr{F}$. An r-uniform hypergraph $\mathscr{F}$ on the set X is *r-partite* if there exists a partition $X = X_1 \cup \cdots \cup X_r$ with $|F \cap X_i| = 1$ for every edge $F \in \mathscr{F}$ and $1 \leqslant i \leqslant r$. $|\mathscr{F}|$ denotes the number of edges of $\mathscr{F}$. In this paper $\log n$ denotes the base 2 logarithm.

1.2. Turán-type hypergraph problems

We let $G^{(r)}(n, m)$ denote the set of r-uniform hypergraphs with n vertices and m edges, and $f^{(r)}(n, p, s)$ is the smallest m such that every member of $G^{(r)}(n, m)$ contains a member of $G^{(r)}(p, s)$. The determination of $f^{(r)}(n, p, s)$ has been a longstanding open problem. Special cases of this problem appeared in [3,4]. For more about Turán-type hypergraph results consult the surveys by Füredi [9] and Sidorenko [13]. In this note we are interested in the growth of $f^{(r)}(n, p, s)$ for fixed values r, p and s.

Brown et al. [2] proved that for $r > k \geqslant 2$ and $s \geqslant 3$ we have

$$f^{(r)}(n, s(r-k)+k, s) = \Theta(n^k).$$

This suggests the following difficult question.

Conjecture 1.

$$f^{(r)}(n, s(r-k)+k+1, s) = o(n^k).$$

This was first established for $r = s = 3$ and $k = 2$ by the celebrated result of Ruzsa and Szemerédi [11]. Then for $s = 3$ and $k = 2$ Erdős et al. [6] extended this result for any r, and they conjectured that it also holds for $k = 2$ and any s. In this direction in [12] we showed that

$$f^{(r)}(n, s(r-2)+2+\lfloor \log s \rfloor, s) = o(n^2). \tag{1}$$

In this note we extend this result for $k > 2$, showing that Conjecture 1 is not far from being true.

Theorem 1. *For all integers $r > k \geqslant 2$ and $s \geqslant 3$,*

$$f^{(r)}(n, s(r-k)+k+\lfloor \log s \rfloor, s) = o(n^k).$$

Thus roughly speaking Conjecture 1 is true with a $\lfloor \log s \rfloor$ term in place of 1. However, it still remains open whether one can replace this term with 1 and prove Conjecture 1.

In addition, by using a recent, deep result of Frankl and Rödl [8] we show that Conjecture 1 is true for $r > k \geqslant 3$ and $s = 4$.

Theorem 2. *For all integers $r > k \geqslant 3$ and $s = 4$,*

$$f^{(r)}(n, 4(r-k)+k+1, 4) = o(n^k).$$

In the next section we provide the tools, then we prove the theorems.

2. Tools

We will use a simple but useful result of Erdős and Kleitman ([7], see also on p. 1300 in [10]).

Lemma 1. *Every k-uniform hypergraph $\mathscr{F}$ contains a k-partite k-uniform hypergraph $\mathscr{H}$ with*

$$\frac{|\mathscr{H}|}{|\mathscr{F}|} \geqslant \frac{k!}{k^k}.$$

We will also need a recent result of Frankl and Rödl. Following their notation from [8], let $A_i = \{a_i, b_i\}$ be pairwise disjoint 2-element sets for $1 \leqslant i \leqslant k$. Define $F_i = \{a_1, \ldots, a_k, b_i\} \setminus \{a_i\}$ and $\mathscr{F}(k) = \{F_1, \ldots, F_k\}$. Let $ex^*(n, \mathscr{F}(k))$ denote $\max |\mathscr{H}|$ such that $\mathscr{H}$ is a k-partite hypergraph on n vertices that is $\mathscr{F}(k)$-free, and $|H \cap H'| \leqslant k-2$ holds for all distinct $H, H' \in \mathscr{H}$. Eq. (3) and Proposition 2.2 of [8] combine to establish the following deep result.

Lemma 2.

$$ex^*(n, \mathscr{F}(4)) = o(n^3).$$

3. Proof of Theorem 1

Let $r > k \geqslant 2$, $s \geqslant 3$, $p = s(r-k) + k + \lfloor \log s \rfloor$. For $k = 2$ we showed that the theorem is true in [12]; thus we may assume $k > 2$.

Assume indirectly that there is a constant $c > 0$ such that, for infinitely many values of n,

$$f^{(r)}(n, p, s) > \lceil cn^k \rceil. \tag{2}$$

From this assumption we will get a contradiction. Eq. (2) means that there exists an r-uniform hypergraph $\mathscr{F}$ with

$$f^{(r)}(n, p, s) - 1 \geqslant \lceil cn^k \rceil \geqslant cn^k$$

edges that does not contain a member of $G^{(r)}(p, s)$, i.e. a set of p vertices spanning at least s edges.

Using the Erdős–Kleitman theorem (Lemma 1) we find an r-partite subhypergraph $\mathscr{H}$ of $\mathscr{F}$ with at least

$$\frac{r!c}{r^r} n^k$$

edges. Let $X_1, \ldots, X_r$ be the vertex classes of this r-partite hypergraph $\mathscr{H}$. Consider the $(k+1)$-uniform hypergraph $\mathscr{H}^*$ which is defined by the removal of $X_1, \ldots, X_{r-k-1}$ from the vertex set of $\mathscr{H}$ and from all edges of $\mathscr{H}$. If a $(k+1)$-edge of $\mathscr{H}^*$ has multiplicity greater than 1, then we keep only one edge. Note that every $(k+1)$-edge has multiplicity less than s. Indeed, otherwise taking a $(k+1)$-edge with multiplicity at least s and s r-edges of $\mathscr{H}$ containing this edge, we get a set of at most

$$s(r-k-1)+k+1 \leqslant s(r-k)+k+\lfloor \log s \rfloor = p$$

vertices that span at least s r-edges, a contradiction. Then if in $\mathscr{H}^*$ we keep only one edge from each multiple $(k+1)$-edge we still have at least

$$\frac{r!c}{r^r(s-1)}\, n^k$$

edges.

Define for every $x_1 \in X_{r-k}, x_2 \in X_{r-k+1}, \ldots, x_{k-2} \in X_{r-3}$ the following hypergraph:

$$\mathscr{H}^*(x_1, \ldots, x_{k-2}) = \{G \setminus \{x_1, \ldots, x_{k-2}\} \mid \{x_1, \ldots, x_{k-2}\} \subset G \in \mathscr{H}^*\}.$$

There are $x_1, \ldots, x_{k-2}$ for which we have

$$|\mathscr{H}^*(x_1, \ldots, x_{k-2})| \geqslant \frac{r!c}{r^r(s-1)}\, n^2.$$

If n is sufficiently large then, by (1), we have a $G^{(3)}(s+2+\lfloor \log s \rfloor, s)$ in this 3-uniform $\mathscr{H}^*(x_1, \ldots, x_{k-2})$. Then in the original $\mathscr{H}$ we have a set of at most

$$s(r-(k+1)) + (k-2) + s + 2 + \lfloor \log s \rfloor = s(r-k) + k + \lfloor \log s \rfloor = p$$

vertices spanning at least s r-edges, a contradiction.

This completes the proof of Theorem 1. □

4. Proof of Theorem 2

Let $r > k \geqslant 3$ and $p = 4(r-k)+k+1$.

Proceeding similarly as above, assume indirectly that there is a constant $c > 0$ such that, for infinitely many values of n,

$$f^{(r)}(n, p, 4) > \lceil cn^k \rceil. \tag{3}$$

From this assumption we will get a contradiction. Eq. (3) means that there exists an r-uniform hypergraph $\mathscr{F}$ with

$$f^{(r)}(n, p, 4) - 1 \geqslant \lceil cn^k \rceil \geqslant cn^k$$

edges that does not contain a member of $G^{(r)}(p, 4)$, i.e. a set of p vertices spanning at least 4 edges.

Similarly as above, first by using Lemma 1 we find an r-partite subhypergraph $\mathscr{H}$ of $\mathscr{F}$ with at least

$$\frac{r!c}{r^r} n^k$$

edges and with partite sets $X_1, \ldots, X_r$. Then we reduce $\mathscr{H}$ to $\{X_{r-k}, \ldots, X_r\}$ to get $\mathscr{H}^*$ with at least

$$\frac{r!c}{3r^r} n^k$$

$(k+1)$-edges. Here, as before, we made use of the fact that any $(k+1)$-edge can be contained in at most three edges of $\mathscr{H}$.

For $k > 3$, similarly as above for every $x_1 \in X_{r-k}, \ldots, x_{k-3} \in X_{r-4}$ we define $\mathscr{H}^*(x_1, \ldots, x_{k-3})$. Again there are $x_1, \ldots, x_{k-3}$ for which we have

$$|\mathscr{H}^*(x_1, \ldots, x_{k-3})| \geqslant \frac{r!c}{3r^r} n^3.$$

For $k > 3$, let $\mathscr{H}^{**}$ be this $\mathscr{H}^*(x_1, \ldots, x_{k-3})$, while for $k = 3$ let $\mathscr{H}^{**} = \mathscr{H}^*$. Thus in both cases

$$|\mathscr{H}^{**}| \geqslant \frac{r!c}{3r^r} n^3.$$

Any 3-edge can be contained in at most three edges of $\mathscr{H}^{**}$, since otherwise we get a set of at most

$$4(r-k-1) + (k-3) + 4 + 3 = 4(r-k) + k < p$$

vertices spanning four edges, a contradiction.

This implies that for any $H' \in \mathscr{H}^{**}$, at most twelve edges $H'' \in \mathscr{H}^{**}$ can exist with $|H' \cap H''| = 3$. Thus we can proceed in a greedy manner, selecting edges of $\mathscr{H}^{**}$ one by one and every time discarding those at most twelve which intersect the new edge in three vertices. Denote the resulting hypergraph by $\overline{\mathscr{H}}$, then

$$|\overline{\mathscr{H}}| \geqslant \frac{r!c}{39r^r} n^3 \qquad (4)$$

and $|H' \cap H''| \leqslant 2$ for all distinct $H', H'' \in \overline{\mathscr{H}}$. Furthermore, $\overline{\mathscr{H}}$ is $\mathscr{F}(4)$-free, since otherwise we get a set of at most

$$4(r-k-1) + (k-3) + 8 = 4(r-k) + k + 1 = p$$

vertices spanning at least four r-edges, a contradiction.

However, then (4) is in contradiction with Lemma 2, if n is sufficiently large.

This completes the proof of Theorem 2. □

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