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On the number of Hamiltonian cycles in Dirac graphs

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Abstract

In response to a question of Bondy, bounds are established on the minimum number of Hamiltonian cycles in all graphs of order n and minimum degree at least $n/2$.

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1. Introduction

1.1. Notations and definitions

For basic graph concepts see the monograph of Bollobás [3].

$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G . (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A \cup B$, and $E \subset A \times B$. For a graph G and a subset U of its vertices, $G|_U$ is the restriction to U of G . $N(v)$ is the set of neighbors of $v \in V$. Hence $|N(v)| = \deg(v) = \deg_G(v)$, the degree of v . $\delta(G)$ stands for the minimum, and $\Delta(G)$ for the maximum degree in G . $K(n_1, n_2, \dots, n_k)$ is the complete k -partite graph with color classes of sizes n_1, n_2, \dots, n_k . P_l denotes the path of length l (counting edges). When A, B are subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B . In particular, we write $\deg(v, U) = e(\{v\}, U)$ for the number of edges from v to U . For non-empty A and B ,

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

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is the *density* of the graph between A and B . In particular, we write $d(A) = d(A, A) = 2|E(G|_A)|/|A|^2$.

Definition 1. The bipartite graph $G = (A, B, E)$ is ε -regular if

$$X \subset A, Y \subset B, |X| > \varepsilon|A|, |Y| > \varepsilon|B| \quad \text{imply} \quad |d(X, Y) - d(A, B)| < \varepsilon,$$

otherwise it is ε -irregular.

We will often say simply that “the pair (A, B) is ε -regular” with the graph G implicit.

Definition 2. (A, B) is (ε, δ) -super-regular if it is ε -regular and

$$\deg(a) > \delta|B| \quad \forall a \in A, \quad \deg(b) > \delta|A| \quad \forall b \in B.$$

1.2. The number of Hamiltonian cycles

Let $h(n, k)$ be the minimum number of Hamiltonian cycles over all labelled graphs of order n and minimum degree k . At one extreme $h(n, n-1) = (n-1)!$. At the other extreme, since the bipartite graph $K(\lceil(n+1)/2\rceil, \lfloor(n-1)/2\rfloor)$ does not have any Hamiltonian cycles, $h(n, k) = 0$ for all $k \leq \lfloor(n-1)/2\rfloor$. A classical result of Dirac [6] states that $h(n, \lceil n/2 \rceil) \geq 1$, so in every graph G of order n with $\delta(G) \geq \lceil n/2 \rceil$ (a *Dirac graph*) there is a Hamiltonian cycle. How many Hamiltonian cycles are there? Nash-Williams [15] proved a lower bound on the number of edge-disjoint Hamiltonian cycles, and as a consequence established that $h(n, \lceil n/2 \rceil) \geq \lfloor \frac{5}{224}n \rfloor$. Bondy (in [5] on page 79 in the Handbook of Combinatorics, and at several conferences, see also Bollobás [4] page 1260 in the same book) asked for an asymptotic estimate of $h(n, \lceil n/2 \rceil)$. Surprisingly there has been no progress on this natural question. In this paper we study this question.

For the upper bound on $h(n, \lceil n/2 \rceil)$, consider the graph $G = K(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$. In case n is odd, add a vertex v_n adjacent to every other vertex. The minimum degree of G is $\lceil n/2 \rceil$. The number of Hamiltonian cycles in $K(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$ is $\lfloor n/2 \rfloor! (\lfloor n/2 \rfloor - 1)!$, and adding vertex v_n for odd n raises this number to $(n-1)\lfloor n/2 \rfloor! (\lfloor n/2 \rfloor - 1)!$.

Our main result is the following.

Theorem 1. *There is a constant $c > 0$ such that*

$$(cn)^n \leq h\left(n, \left\lceil \frac{n}{2} \right\rceil\right) \leq (n-1) \left\lfloor \frac{n}{2} \right\rfloor! \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right)! \quad (1)$$

Note that here for sufficiently large n the upper bound is less than or equal to

$$3\pi n \left(\frac{n}{2e} \right)^n.$$

We conjecture that this upper bound is close to the truth, so

$$\lim_{n \rightarrow \infty} \frac{(h(n, \lceil \frac{n}{2} \rceil))^{1/n}}{n} = \frac{1}{2e}.$$

Theorem 1 shows that if this limit exists it is positive.

In fact, perhaps this is the right order of magnitude for graphs with the other well-known weaker degree conditions guaranteeing Hamiltonicity, such as Ore, Pósa and Chvátal (see [5]).

2. The main tools

In the proof the Regularity Lemma [20] plays a central role. Here we will use the following variation of the lemma. For a proof, see [14].

Lemma 2 (Regularity Lemma—degree form). *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex-set V into $l + 1$ sets (so-called clusters) V_0, V_1, \dots, V_l , and there is a subgraph $G' = (V, E')$ with the following properties:*

- $l \leq M$,
- $|V_0| \leq \varepsilon|V|$,
- all clusters $V_i, i \geq 1$, are of the same size $L \leq \lceil \varepsilon|V| \rceil$.
- $\deg_{G'}(v) > \deg_G(v) - (d + \varepsilon)|V|$ for all $v \in V$,
- $G'|_{V_i} = \emptyset$ (V_i are independent in G'),
- all pairs $G'|_{V_i \times V_j}, 1 \leq i < j \leq l$, are ε -regular, each with a density 0 or exceeding d .

This form can easily be obtained by applying the original Regularity Lemma (with a smaller value of ε), adding to the exceptional set V_0 all clusters incident to many irregular pairs, and then deleting all edges between any other clusters where the edges either do not form a regular pair or they do but with a density at most d .

Recently we have developed a general tool, the Blow-up Lemma, for applications of the Regularity Lemma (see [10] for the original, [12] for an algorithmic version and [16,17] for two alternate proofs). Here we use the following very special case of a counting version of the Blow-up Lemma.

Lemma 3. *Given $\delta > 0$ there exists an $\varepsilon = \varepsilon(\delta) > 0$ such that the following holds. Let $G = (A, B, E)$ be an (ε, δ) -super-regular pair with $|A| = |B| = n/2$ and $x \in A, y \in B$. Then the number of Hamiltonian paths in G starting at x and ending at y is at least $(\varepsilon n)^n$.*

A proof of the counting version of the Blow-up Lemma (with a much better constant) can be found in [16, p. 446] (see also [2,18]).

Finally, we are going to use the following easy consequence of Dirac's theorem.

Lemma 4. *Let G be a graph on n vertices. If $\delta(G) \geq (n/2) - x$ for some natural number x , then there is a matching in G which covers all but at most $2x + 1$ exceptional vertices.*

This follows from adding $2x$ extra vertices to G adjacent to all other vertices. The new graph is a Dirac graph and admits a Hamiltonian cycle, and a matching which covers all but at most one vertex.

3. Outline of the proof

In a series of papers [8,9,10,11,12,13], a general method based on the Regularity Lemma and the Blow-up Lemma for embedding problems in dense graphs was developed. In this paper for the lower bound in (1) we use a counting version (Lemma 3) of this method.

Let us consider a graph G of order n with

$$\delta(G) \geq \frac{n}{2}. \quad (2)$$

We will assume throughout the paper that n is sufficiently large. We will use the following main parameters.

$$\varepsilon \ll d \ll \alpha \ll 1, \quad (3)$$

where d depends on α , ε depends on d and α , and $a \ll b$ means that a is sufficiently small compared to b . For simplicity we do not compute the actual dependencies, although it could be done.

First in the next section, in the non-extremal part of the proof, we assume that the following extremal condition does not hold for our graph G . We show later in Section 5 that Theorem 1 is true in the extremal case as well.

Extremal Condition (EC): There exist (not necessarily disjoint) $A, B \subset V(G)$ such that

- $|A| = |B| = \lfloor n/2 \rfloor$, and
- $d(A, B) < \alpha$.

We apply Lemma 2 for G with ε and d as in (3). We get a partition of $V(G')$ into clusters V_0, V_1, \dots, V_l . We define the following *reduced graph* G_r : The vertices of G_r are the clusters $V_i, 1 \leq i \leq l$, and we have an edge between two clusters if they form an ε -regular pair in G' with density exceeding d . Since in G' , $\delta(G') \geq (\frac{1}{2} - (d + \varepsilon))n$, an easy calculation shows that in G_r we have

$$\delta(G_r) \geq \left(\frac{1}{2} - 3d \right) l. \quad (4)$$

Indeed, because the neighbors of u in G' can only be in V_0 and in the clusters which are neighbors of V_i in G_r , then for a $V_i, 1 \leq i \leq l$ we have:

$$\left(\frac{1}{2} - (d + \varepsilon) \right) nL \leq \sum_{u \in V_i} \deg_{G'}(u) \leq \varepsilon nL + \deg_{G_r}(V_i)L^2.$$

From this we get inequality (4):

$$\deg_{G_r}(V_i) \geq \left(\frac{1}{2} - d - 2\varepsilon \right) \frac{n}{L} \geq \left(\frac{1}{2} - 3d \right) l.$$

Applying Lemma 4 we find a matching M in G_r covering all but at most $6dl + 1$ clusters. We add the vertices of these exceptional clusters to the exceptional set V_0 ,

so that

$$|V_0| \leq 6dlL + \varepsilon n + L \leq 7dn. \tag{5}$$

For each edge of the matching M , say the edge joining clusters V_i and V_j , we want to use Lemma 3 to extract a large number of Hamiltonian paths in the subgraph of G induced by $V_i \cup V_j$ to be spliced into the Hamiltonian cycles of G . However, we must first remove some vertices from V_i and V_j to achieve super-regularity, and then remove more vertices so $|V_i| = |V_j|$. To absorb these extra vertices, we first modify M in G_r to a covering by vertex disjoint triangles and a matching, then the removed vertices are added to the triangles. Indeed, Lemma 3 will be applied in each edge of the matching of the cover, and in the first two clusters of each triangle. But for this purpose, first we are going to connect these pairs by short connecting paths that will be on every Hamiltonian path, then we will take care of the various exceptional vertices and make some adjustments. The third cluster in the triangles will help in this adjustment, namely to balance the number of remaining vertices in the two other clusters of the triangle.

4. The non-extremal case

4.1. Modifying the matching

For any Dirac graph G we apply Lemma 2 with ε and d as in (3), and then we obtain the reduced graph G_r and the matching M described in the previous section. We assume for simplicity that $(\sqrt{d}/2)l$ is an integer. We remove any $(\sqrt{d}/2)l$ edges (and thus $\sqrt{d}l$ clusters) from M to yield matching M' . Let us denote $|M'|$ by s . Since we have

$$\frac{l}{2} \geq |M| \geq \frac{l - 6dl - 1}{2},$$

and we removed $(\sqrt{d}/2)l$ edges from M to get M' , we have

$$\left(\frac{1}{2} - \frac{\sqrt{d}}{2}\right)l \geq s \geq \frac{l - 6dl - 1 - \sqrt{d}l}{2} \geq \left(\frac{1}{2} - \sqrt{d}\right)l, \tag{6}$$

if d is sufficiently small. Our goal in this section is to show that by slightly changing M' and by redistributing the removed clusters, we can get a vertex disjoint cover in which we have $\sqrt{d}l$ triangles and a matching of size $s - \sqrt{d}l$. We will denote this covering by M_1, M_2, \dots, M_s , where $M_i, 1 \leq i \leq s' = \sqrt{d}l$ is a triangle and $M_i, s' < i \leq s$ is an edge, and we denote the clusters in M_i by V_1^i, V_2^i (V_3^i if $1 \leq i \leq s'$).

We redistribute the $\sqrt{d}l$ removed clusters one-by-one. Let us consider the next removed cluster C . If there is an edge M_i in the current cover such that we have $\{C, V_1^i\}, \{C, V_2^i\} \in E(G_r)$, then we just add C to M_i , we have one more triangle and we can move to the next removed cluster. Thus we may assume that $|N_{G_r}(C) \cap M_i| \leq 1$

for every edge M_i in the current cover. Let T be the set of edges M_i in the current cover for which $|N_{G_r}(C) \cap M_i| = 1$. Then (3), (4), (6) and an easy calculation show that $|T| \geq (1 - d^{1/3})s$. Indeed, otherwise let us count the neighbors of C , and we get a contradiction. C has exactly 1 neighbor in each edge of T (fewer than $(1 - d^{1/3})s$ neighbors), at most 3 neighbors in each triangle (fewer than $3\sqrt{dl}$ neighbors), and there are at most $6dl + 1 + \sqrt{dl}$ clusters not covered by the current cover. But then we get a contradiction from (2), (4) and (6) since

$$\begin{aligned} \deg_{G_r}(C) &< 6dl + 1 + \sqrt{dl} + 3\sqrt{dl} + (1 - d^{1/3})s \leq (7d + 4\sqrt{d})l \\ &+ (1 - d^{1/3}) \left(\frac{1}{2} - \frac{\sqrt{d}}{2} \right) l \\ &= \left(\frac{1}{2} - \frac{d^{1/3}}{2} + \frac{7\sqrt{d}}{2} + \frac{d^{5/6}}{2} + 7d \right) l < \left(\frac{1}{2} - \frac{d^{1/3}}{4} \right) l \\ &< \left(\frac{1}{2} - 3d \right) l \leq \deg_{G_r}(C). \end{aligned}$$

Let S be the set of clusters in edges of T which are not neighbors of C . We could exchange C with any cluster of S in the corresponding edge of T . As above, if we have a $C' \in S$ and edges M_i, M_j such that $C' \in M_i$ and $\{C', V_1^j\}, \{C', V_2^j\} \in E(G_r)$, then again we are done, since we replace C' with C in M_i and add C' to M_j , and thus we have one more triangle. Hence we may assume that there is no C' with this property.

Let A be the union of the clusters in S , along with enough other vertices to satisfy $|A| = \lfloor n/2 \rfloor$. Since EC does not hold, $d_G(A) = d_G(A, A) \geq \alpha$. This and $d \ll \alpha$ clearly imply that there exists a $C_1 \in S$ such that

$$\deg_{G_r}(C_1, S) \geq \frac{\alpha}{2} \frac{l}{2}.$$

Say $M_i = \{C_1, C'_1\}$. Consider an arbitrary $C_2 \in N_{G_r}(C_1, S)$ with $\{C_2, C'_2\} \in M'$. We replace C_2 with C in $\{C_2, C'_2\}$, we replace C'_1 with C_2 in M_i , and now C'_1 plays the role of C . Then again, if we have an edge M_j such that $\{C'_1, V_1^j\}, \{C'_1, V_2^j\} \in E(G_r)$, then we are done, so we may assume that there is no edge M_j with this property. Then similarly as above for C , this implies that the number of edges M_i in the current cover for which $|N_{G_r}(C'_1) \cap M_i| = 1$ is at least $(1 - d^{1/3})s$.

Now if we have an arbitrary $C_2 \in N_{G_r}(C_1, S)$ with $\{C_2, C'_2\} \in M'$, then we have $\{C_1, C'_2\}, \{C_2, C'_1\} \notin E(G_r)$, since otherwise we have two edges to the same edge of M' from C_1 or from C_2 , and we could increase the number of triangles in M' .

Finally using $d \ll \alpha$, the above clearly implies that we can pick a $\{C_2, C'_2\} \in M'$ such that $\{C_1, C_2\} \in E(G_r)$, $\{C_1, C'_2\}, \{C_2, C'_1\} \notin E(G_r)$ and $\{C'_1, C'_2\} \in E(G_r)$. Then replace C'_2 with C_1 in the edge of M' , and $M_i = \{C, C'_1, C'_2\}$ yields one more triangle again. By repeating this procedure we obtain a sequence M_1, M_2, \dots, M_s where $M_i, 1 \leq i \leq s$, are triangles and the others are edges.

4.2. Connecting the M_i -s

For $1 \leq i \leq s$ we pick typical vertices u_i and v_i in the following way:

- $u_i \in V_1^i$ and $\deg(u_i, V_2^i) \geq (d - \varepsilon)|V_2^i|$,
- $v_i \in V_2^i$ and $\deg(v_i, V_1^i) \geq (d - \varepsilon)|V_1^i|$.

Note that ε -regularity implies that most vertices satisfy this in V_1^i and V_2^i . We use the following simple consequence of the fact that EC does not hold.

Claim 5. *If EC does not hold in a Dirac graph, then for a pair of distinct vertices u and v we have at least $(\alpha/2)/(n/2)$ vertex disjoint P_3 -s connecting u and v .*

Indeed, to get A in EC we keep $\lfloor n/2 \rfloor$ neighbors of u , to get B we keep $\lfloor n/2 \rfloor$ neighbors of v , and then $d(A, B) \geq \alpha$ implies the claim.

Using the claim we find vertex disjoint P_3 -s connecting u_i and v_{i+1} in G (where $v_{s+1} = v_1$). This is possible since the number of these paths is a constant, s . These paths are the connecting paths between M_i and M_{i+1} , they will be on all the Hamiltonian cycles that we are counting for the lower bound in (1), and the vertices on these paths between u_i and v_{i+1} are removed from G . However, note again that since the number of removed vertices is a constant, this will be absorbed by the constant in the lower bound in (1). We may assume that the number of vertices remaining in G is even by inserting a vertex into the connecting path between u_1 and v_2 if necessary.

4.3. Adjustments and the handling of the exceptional vertices

We already have a set V_0 of exceptional vertices in G . We add some more vertices to V_0 to achieve super-regularity across all the edges of M_1, \dots, M_s . From a cluster V_j^i in M_i we remove all vertices v for which there exists a $j' \neq j$ such that

$$\deg(v, V_{j'}^i) \leq (d - \varepsilon)|V_{j'}^i|.$$

ε -regularity guarantees that at most $2\varepsilon|V_j^i| \leq 2\varepsilon L$ such vertices exist in each cluster V_j^i (note that we need the factor of 2 because of the triangles).

At this point we may have a small discrepancy among the number of remaining vertices in the clusters. By removing extra vertices from certain clusters (and adding them to the exceptional set V_0) we achieve that each cluster has exactly L' vertices where $2|L'|$. From (5), V_0 had at most $7dn$ vertices, and adding at most another $2\varepsilon Ll \leq 2\varepsilon n$ vertices, we still have

$$|V_0| \leq 8dn. \tag{7}$$

Next, since we are looking for spanning subgraphs, namely Hamiltonian cycles, we have to take care of the vertices in V_0 . We are going to redistribute these vertices among the M_i -s, $1 \leq i \leq s$. We group the vertices in V_0 into pairs of vertices (note that $|V_0|$ is even), and we associate with each cluster C , $C \neq V_0$, an initially empty subset $H(C)$ to receive the distributed vertices. Vertices added to $H(C)$ are also added to C .

Consider the first pair of vertices v_1, v_2 in V_0 . First we show that with some exchanges and maintaining super-regularity we can achieve that the two vertices we have to redistribute both came from the same cluster.

Let us assume first that there is an $M_i, i > s'$ and a cluster in M_i , say V_1^i without loss of generality, such that

$$\deg(v_j, V_1^i) \geq (d - \varepsilon)|V_1^i|, j \in \{1, 2\}. \quad (8)$$

(Here and in the remainder of this section we keep the same names for the clusters although the contents may change as we process pairs in V_0 .) In this case we may add v_1 and v_2 to $H(V_2^i)$, and now remove any two vertices from V_2^i and rename them v_1 and v_2 . Note that (8) implies that with this exchange we maintained super-regularity in M_i . The situation is similar if we have V_2^i instead of V_1^i in (8).

Let us denote by $S_j, j \in \{1, 2\}$, the sets of those clusters C for which $C \in M_i$ for some $i > s'$ and

$$\deg(v_j, C) \geq (d - \varepsilon)|C|. \quad (9)$$

Note that

$$|S_j| \geq \left(\frac{1}{2} - d^{1/3}\right)l \quad \text{for } j = 1, 2. \quad (10)$$

Indeed, if (10) were not true we get a contradiction by counting the neighbors of v_j . v_j can have at most $8dn$ neighbors in V_0 (using (7)), fewer than $3\sqrt{dn}$ neighbors in the triangles, at most $(d - \varepsilon)n$ neighbors in clusters in $M_i, i > s'$ for which (9) is not true, and finally at most $(\frac{1}{2} - d^{1/3})n$ in the clusters in $M_i, i > s'$ for which (9) is true. Hence altogether,

$$\deg(v_j) < 8dn + 3\sqrt{dn} + dn + \left(\frac{1}{2} - d^{1/3}\right)n \leq \left(\frac{1}{2} - \frac{d^{1/3}}{2}\right)n < \frac{n}{2},$$

a contradiction with (2).

Denote by \bar{S}_j the “complement” clusters in M_i , namely if $V_1^i \in S_j$, then $V_2^i \in \bar{S}_j$, and if $V_2^i \in S_j$, then $V_1^i \in \bar{S}_j$. Consider the set of vertices of G in \bar{S}_1 and in S_2 . Let A and B be arbitrary sets of cardinality $\lfloor n/2 \rfloor$ such that $\bigcup_{C \in \bar{S}_1} C \subset A$ and $\bigcup_{C \in S_2} C \subset B$. Since EC does not hold we have $d_G(A, B) \geq \alpha$. This fact, (3) and (10) imply that there are clusters C_1 and C_2 such that $C_1 \in \bar{S}_1$, $C_2 \in S_2$ and $\{C_1, C_2\} \in E(G_r)$. Consider a vertex $v'_1 \in C_1$ with $\deg(v'_1, C_2) \geq (d - \varepsilon)|C_2|$ (from ε -regularity most vertices are such in C_1). Exchange v_1 with v'_1 , and in fact, we add v_1 to $H(C_1)$. This exchange does not hurt super-regularity since $C_1 \in \bar{S}_1$. But now we have

$$\deg(v'_1, C_2) \geq (d - \varepsilon)|C_2| \quad \text{and} \quad \deg(v_2, C_2) \geq (d - \varepsilon)|C_2|.$$

Thus we are back to the special case discussed above, we may exchange v'_1 and v_2 with two vertices from the other cluster of M_i , if $C_2 \in M_i$.

Hence now we may assume that v_1 and v_2 came from the same cluster C . For this cluster C and for every $M_i, 1 \leq i \leq s$, we determine a label

$$l_i = |N_{G_i}(C) \cap M_i|.$$

The number of M_i -s, $1 \leq i \leq s$ for which $l_i \geq 2$ is at least $d^{2/3}s$. In fact, otherwise let us count the neighbors of C and we will get a contradiction from (4). C can have at most 3 neighbors in fewer than $d^{2/3}s$ triangles, and it has at most 1 neighbor in the remaining triangles, thus altogether C can have fewer than $\sqrt{dl} + 2d^{2/3}s$ neighbors in the triangles. Similarly C can have fewer than $s - \sqrt{dl} + d^{2/3}s$ neighbors in the edges, and finally there are at most $6dl + 1$ clusters not covered by the triangles and edges. Hence, using (3), (4) and (6) we get a contradiction

$$\begin{aligned} \deg_{G_r}(C) &< 6dl + 1 + \sqrt{dl} + 2d^{2/3}s + (s - \sqrt{dl}) + d^{2/3}s \\ &= s + 6dl + 1 + 3d^{2/3}s \\ &\leq \left(\frac{1}{2} - \frac{\sqrt{d}}{2}\right)l + 4d^{2/3} \left(\frac{1}{2} - \frac{\sqrt{d}}{2}\right)l < \left(\frac{1}{2} - \frac{\sqrt{d}}{2} + 2d^{2/3}\right)l \\ &< \left(\frac{1}{2} - 3d\right)l \leq \deg_{G_r}(C). \end{aligned}$$

Pick an M_i with $l_i \geq 2$. In case $i > s'$, we add v_1 to $H(V_1^i)$ and v_2 to $H(V_2^i)$. If $1 \leq i \leq s'$ and $l_i = 2$, then we add v_1 and v_2 to H of the third cluster in M_i that is not the neighbor of C in G_r . Finally, if $1 \leq i \leq s'$ and $l_i = 3$, then we add v_1 and v_2 to H of any cluster in M_i .

Next we want to handle subsequent pairs of vertices in V_0 . Unfortunately, because $|V_0|$ may be quite large, we cannot just repeat this procedure for all vertices in V_0 , since we might violate super-regularity. First of all, (7) and the fact that we can add any pair from V_0 to at least $d^{2/3}s$ M_i -s (and similarly for the exchanging) implies that we may achieve that during the whole process $|H(C)| \leq d^{1/4}|C|$ for any cluster C . However, even this is significantly larger than $\varepsilon|C|$, and thus we might violate ε -regularity. Therefore we have to do periodic updating in each M_i , $1 \leq i \leq s$. We will have two kinds of updating.

Update 1: When $|H(V_j^i)| \geq \varepsilon^2|V_j^i|$ becomes true for some i and j , we do the first kind of updating in this M_i , namely we eliminate the added vertices in M_i with the following procedure.

First let us assume that $i > s'$, so $M_i = \{V_1^i, V_2^i\}$ is an edge in the cover. Pick sets A and B in the following way.

- $H(V_1^i) \subset A \subset V_1^i$, $H(V_2^i) \subset B \subset V_2^i$,
- $|A| = |B| = \lfloor \sqrt{\varepsilon}|V_1^i| \rfloor$,
- $A \setminus H(V_1^i)$ and $B \setminus H(V_2^i)$ are chosen randomly from $V_1^i \setminus H(V_1^i)$ and $V_2^i \setminus H(V_2^i)$.

Then with high probability the pair (A, B) is also super-regular. More precisely:

Claim 6. *With high probability $G|_{A \times B}$ is (ε', d') -super-regular for $d' = d/2$ and for some $\varepsilon' \ll d'$.*

This well-known claim follows from the following two facts.

Fact 7. *With high probability $G|_{A \times B}$ is ε' -regular.*

Fact 8. *With high probability $\delta(G|_{A \times B}) > d'|A|$.*

Fact 8 follows from Chernoff's bound and the fact that M_i is also super-regular, so

$$\delta(G|_{M_i}) \geq (d - \varepsilon)|V_1^i|.$$

For Fact 7, observe, that with high probability (about $1 - O(|A|e^{-\varepsilon|A|})$), apart from at most $4\varepsilon|A|^2$ exceptional pairs in A , for a pair $\{x, x'\} \subset A$ we have

$$|N_G(x) \cap N_G(x') \cap B| \geq (d - \varepsilon)^2|B|.$$

This so-called quasi-random property is well-known to imply Fact 7 (see [19,1,7]).

Since Claim 6 is true, we can apply Lemma 3 for $G|_{A \times B}$, where x is an arbitrary neighbor of v_i in A and y is an arbitrary vertex of B . This extends the connecting path between M_{i-1} and M_i by a Hamiltonian path in $G|_{A \times B}$. By Lemma 3 the number of ways we can do this is at least $(\varepsilon'2|A|)^{2|A|}$. y is the new v_i , the endpoint of the connecting path between M_{i-1} and M_i .

In case $1 \leq i \leq s'$, we do the following. First we do the above procedure in $\{V_1^i, V_2^i\}$ to eliminate $H(V_1^i)$ and $H(V_2^i)$. Then again we repeat the procedure, but this time in (V_2^i, V_3^i) to eliminate $H(V_3^i)$ (note that $H(V_2^i)$ was already eliminated in the previous step). Thus again all the added vertices are eliminated.

Update 2: When we eliminated $(d/2)|V_j^i|$ vertices from V_j^i for some i and j , we do the second kind of updating, namely we remove the vertices from M_i that violate super-regularity.

From a cluster V_j^i in M_i we remove all vertices v for which there exists a $j' \neq j$ such that

$$\deg(v, V_{j'}^i) \leq (d - \varepsilon)|V_{j'}^i|.$$

If $i > s'$, we also remove some extra vertices to make sure that we have the same number of vertices left in the two clusters of M_i . If $1 \leq i \leq s'$, we remove one more vertex, if necessary, to make sure that the number of remaining vertices in M_i is even. ε -regularity guarantees that during the whole process in Update 2 we added at most $\sqrt{\varepsilon}n$ ($\ll dn$) new vertices to V_0 .

From the above we get that the procedure goes through for all the pairs of vertices from V_0 , and that during the whole process, we eliminated at most $(d^{1/4})^{1/4}|C| = d^{1/16}|C|$ vertices from each cluster C .

4.4. Filling in the Hamiltonian cycles

We now have all but a constant number of vertices belonging to clusters of a covering M_1, M_2, \dots, M_s , where M_i , $1 \leq i \leq \sqrt{dl}$ are triangles of G_r , and M_i , $\sqrt{dl} < i \leq s$, are edges of G_r . The pairs of clusters in every M_i are (ε', d') -super-regular clusters of G . Either $2l$ or $2l + 1$ vertices were used to connect the M_i -s.

In each $M_i, i > \sqrt{dl}$ we have $|V_1^i| = |V_2^i|$, and the bipartite graph (V_1^i, V_2^i) is (ε', d') -super-regular with $\varepsilon' \ll d'$. By Lemma 3, there are at least $(c_i |V_1^i|)^{2|V_1^i|}$ Hamiltonian paths in (V_1^i, V_2^i) between u_i and v_i . For $M_i, 1 \leq i \leq \sqrt{dl}$, we might have a discrepancy among the remaining number of vertices in the three clusters. We distribute the vertices of V_3^i among V_1^i and V_2^i such that now $|V_1^i| = |V_2^i|$ (note that the number of remaining vertices was even in M_i). Furthermore, the resulting bipartite graph (V_1^i, V_2^i) is still (ε', d') super-regular with $\varepsilon' \ll d'$. Again Lemma 3 assures us that there must be at least $(c_i |V_1^i|)^{2|V_1^i|}$ Hamiltonian paths in (V_1^i, V_2^i) between u_i and v_i , and this finishes the non-extremal case.

5. The extremal case

First we assume that we have the following special case.

Case 1: There is a partition $V(G) = A_1 \cup A_2$ with $|A_1| = \lfloor n/2 \rfloor$ and $d(A_1) < \alpha^{1/3}$.

Note that in this case from (2) we also have $d(A_1, A_2) > 1 - \alpha^{1/3}$. Thus roughly speaking in this case we have very few edges in $G|_{A_1}$, and we have an almost complete bipartite graph between A_1 and A_2 .

A vertex $v \in A_i, i \in \{1, 2\}$, is called *bad* if it is not connected to most of the vertices in the other set, more precisely if we have

$$\deg(v, A_{i'}) \leq (1 - \alpha^{1/9})|A_{i'}|, \{i, i'\} = \{1, 2\}. \quad (11)$$

Note that since $\delta(G) \geq n/2$, this implies that if $v \in A_i$ is bad, then

$$\deg(v, A_i) \geq \alpha^{1/9}|A_i|.$$

Furthermore, since $d(A_1, A_2) > 1 - \alpha^{1/3}$, the number of bad vertices in A_i is at most

$$\alpha^{2/9}|A_i| (\ll \alpha^{1/9}|A_i|).$$

We also define a special type of bad vertex; a vertex $v \in A_i, i \in \{1, 2\}$, is *exceptionally bad* if it is connected to very few vertices in the other set, that is

$$\deg(v, A_{i'}) \leq \frac{\alpha^{1/9}}{2}|A_{i'}|, \{i, i'\} = \{1, 2\}.$$

Note that if a vertex is exceptionally bad, then clearly it is bad.

Let us assume first that we have no exceptionally bad vertices. If $n \equiv 1 \pmod{2}$, then $|A_2| = |A_1| + 1$. Since $\delta(G|_{A_2}) \geq 1$, it is easy to see that we can find an edge e in $G|_{A_2}$ such that the two endpoints of e have at least $\frac{\alpha^{1/9}}{4}n$ common neighbors in A_1 . Indeed, to find this e , take an arbitrary vertex $w \in A_2$ with

$$\deg(w, A_1) \geq (1 - \alpha^{1/3})|A_1|$$

(the existence of w is guaranteed by $d(A_1, A_2) \geq 1 - \alpha^{1/3}$) and e is an arbitrary edge in $G|_{A_2}$ incident to w (there must be an edge incident to w , since $\delta(G|_{A_2}) \geq 1$). We contract e into a vertex, where the neighbors of this new vertex are the common

neighbors of the two endpoints of e . Consider the bipartite graph $G|_{A_1 \times A_2}$. This is an almost complete bipartite graph between two equal color classes, where the number of bad vertices is much smaller than their degree, and thus the lower bound in (1) clearly holds. When the Hamiltonian cycle in this bipartite graph goes through w , then in G we traverse the edge e .

Thus, we have to eliminate only the exceptionally bad vertices, the other bad vertices do not cause any further complications. We show first that we may assume that we have either no bad vertices in $A_i, i \in \{1, 2\}$, or there are no exceptionally bad vertices in $A_{i'}, \{i', i\} = \{1, 2\}$. Indeed, otherwise we could exchange a bad vertex in A_i with an exceptionally bad vertex in $A_{i'}$ and this way we can decrease the number of bad vertices in A_i , since the exceptionally bad vertex of $A_{i'}$ will not be bad in A_i . By iterating this procedure we can achieve that either we have no more bad vertices in A_i , or there are no more exceptionally bad vertices left in $A_{i'}$. Thus we can have exceptionally bad vertices only in at most one of the sets A_1 and A_2 . Assume that we have exceptionally bad vertices in A_1 . Then we have no bad vertices in A_2 . We remove the exceptionally bad vertices from A_1 and we add them to A_2 . Note that these added vertices will not be bad in A_2 . For simplicity we still denote the sets by A_1 and A_2 . Let $|A_2| = |A_1| + x$. We have the following.

Claim 9. $G|_{A_2}$ contains at least $(x/4)^x$ matchings of size x .

Indeed, we have

$$\delta(G|_{A_2}) \geq \frac{n}{2} - \frac{n-x}{2} = \frac{x}{2}.$$

We choose the edges of the matching one-by-one and we show that for the next edge we always have at least $x/4$ choices. For the first edge we pick an arbitrary vertex u_1 , and we have at least $x/2 > x/4$ choices for v_1 from the neighbors of u_1 in $G|_{A_2}$. Suppose we have already chosen the edges $\{u_1, v_1\}, \dots, \{u_i, v_i\}, 1 \leq i < x$ in the matching. We show that we have at least $x/4$ choices for $\{u_{i+1}, v_{i+1}\}$. In fact, we must have a vertex u_{i+1} with

$$\left| N_{G|_{A_2}} \left(u_{i+1}, A_2 \setminus \bigcup_{j=1}^i \{u_j, v_j\} \right) \right| \geq \frac{x}{4},$$

since otherwise we would have at least

$$\left(\frac{n}{2} - 2x \right) \frac{x}{4} \geq \frac{nx}{16}$$

edges between $\bigcup_{j=1}^i \{u_j, v_j\}$ and $A_2 \setminus \bigcup_{j=1}^i \{u_j, v_j\}$, and this is impossible since we had no bad vertices in A_2 . Then we pick this u_{i+1} , and we have at least $x/4$ choices for v_{i+1} proving the claim.

We contract the edges in the matching of size x and denote the resulting set by A'_2 . Similarly as above, the bipartite graph $G|_{A_1 \times A'_2}$ is an almost complete bipartite graph

between two equal color classes. Thus in G we have at least

$$(c(n-x))^{n-x} \left(\frac{x}{4}\right)^x \geq (c'n)^n$$

Hamiltonian cycles for some constant $c' > 0$, proving the lower bound in (1) for this case.

Case 2: Assume next that we have a partition $V(G) = A_1 \cup A_2$ with $|A_1| = \lfloor n/2 \rfloor$ and $d(A_1, A_2) < \alpha^{1/3}$. Thus, roughly speaking, $G|_{A_1}$ and $G|_{A_2}$ are almost complete and the bipartite graph between A_1 and A_2 is sparse.

Again we define *bad* vertices $v \in A_i, i \in \{1, 2\}$, as

$$\deg(v, A_{i'}) \geq \alpha^{1/9} |A_{i'}|, \{i, i'\} = \{1, 2\}.$$

Note that again the number of bad vertices in A_i is at most $\alpha^{2/9} |A_i|$. We remove the bad vertices from each set and add them to the set where they have more neighbors. We still denote the sets by A_1 and A_2 . Say $|A_1| \leq |A_2|$.

We show that we can find two vertex disjoint edges $\{u_1, v_1\}, \{u_2, v_2\}$ in $G|_{A_1 \times A_2}$. This is trivial if $|A_1| < |A_2|$, since then for every $u \in A_1$ we have $\deg(u, A_2) \geq 2$. Thus we may assume that $|A_1| = |A_2|$. But then for every $u \in A_1$ we have $\deg(u, A_2) \geq 1$ and for every $v \in A_2$ we have $\deg(v, A_1) \geq 1$, and thus again we can pick the two edges.

Since $G|_{A_1}$ and $G|_{A_2}$ are almost complete graphs, where the number of bad vertices is much smaller than their degree, we are done again because we have sufficiently many Hamiltonian paths in $G|_{A_1}$ connecting u_1 and u_2 and in $G|_{A_2}$ connecting v_1 and v_2 .

Case 3: Assume finally that the extremal case EC holds, so we have $A, B \subset V(G)$, $|A| = |B| = \lfloor n/2 \rfloor$ and $d(A, B) < \alpha$. We have three possibilities.

- $|A \cap B| < \sqrt{\alpha}n$. The statement follows from Case 2. Indeed, let $A_1 = A$, $A_2 = V(G) \setminus A_1$, then clearly $d(A_1, A_2) < \alpha^{1/3}$ if $\alpha \leq 1$ holds.
- $\sqrt{\alpha}n \leq |A \cap B| < (1 - \sqrt{\alpha})n/2$. This case is not possible under the given conditions.

In fact, otherwise we would get

$$\begin{aligned} |A \cap B| \frac{n}{2} &\leq \sum_{u \in A \cap B} \deg_G(u) = \sum_{u \in A \cap B} \deg_G(u, A \cup B) \\ &\quad + \sum_{u \in A \cap B} \deg_G(u, V(G) \setminus (A \cup B)) \\ &\leq 2\alpha n^2 + |A \cap B| (|A \cap B| + 1), \end{aligned}$$

or

$$|A \cap B| \left(\frac{n}{2} - |A \cap B| - 1 \right) \leq 2\alpha n^2,$$

a contradiction under the given conditions.

- $|A \cap B| \geq (1 - \sqrt{\alpha})n/2$. The statement follows from Case 1 by choosing $A_1 = A$, $A_2 = V(G) \setminus A_1$, and then $d(A_1) < \alpha^{1/3}$.

This finishes the extremal case and the proof of the lower bound in (1).

References

- [1] N. Alon, R. Duke, H. Leffman, V. Rödl, R. Yuster, The algorithmic aspects of the Regularity Lemma, *J. Algorithms* 16 (1994) 80–109.
- [2] N. Alon, V. Rödl, A. Ruciński, Perfect matchings in ε -regular graphs, *Electronic J. Combin.* 5(1) (1998) R13.
- [3] B. Bollobás, *Extremal Graph Theory*, Academic Press, London, 1978.
- [4] B. Bollobás, Extremal graph theory, in: R.L. Graham, M. Grottschel, L. Lovasz (Eds.), *Handbook of Combinatorics*, Elsevier Science, Amsterdam, 1995, pp. 1231–1292.
- [5] J.A. Bondy, Basic graph theory: Paths and circuits, in: *Handbook of Combinatorics*, Elsevier Science, Amsterdam, 1995, pp. 3–110.
- [6] G.A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* 2 (1952) 68–81.
- [7] Y. Kohayakawa, Szemerédi’s Regularity Lemma for sparse graphs, in: F. Cucker, M. Schub (Eds.), *Foundations of Computational Mathematics*, Springer, Berlin, 1997.
- [8] J. Komlós, G.N. Sárközy, E. Szemerédi, Proof of a packing conjecture of Bollobás, *Combinatorics, Probab. Comput.* 4 (1995) 241–255.
- [9] J. Komlós, G.N. Sárközy, E. Szemerédi, On the square of a Hamiltonian cycle in dense graphs, *Random Struct. Algorithms* 9 (1996) 193–211.
- [10] J. Komlós, G.N. Sárközy, E. Szemerédi, Blow-up Lemma, *Combinatorica* 17 (1997) 109–123.
- [11] J. Komlós, G.N. Sárközy, E. Szemerédi, On the Pósa-Seymour conjecture, *J. Graph Theory* 29 (1998) 167–176.
- [12] J. Komlós, G.N. Sárközy, E. Szemerédi, An algorithmic version of the Blow-up Lemma, *Random Struct. Algorithms* 12 (1998) 297–312.
- [13] J. Komlós, G.N. Sárközy, E. Szemerédi, Proof of the Seymour conjecture for large graphs, *Ann. Combin.* 2 (1998) 43–60.
- [14] J. Komlós, E. Szemerédi, Szemerédi’s Regularity Lemma and its applications in Graph Theory, in: D. Miklós, V.T. Sós, T. Szőnyi (Eds.), *Combinatorics, Paul Erdős is Eighty, Vol. 2*, Bolyai Society Mathematical Studies, Budapest, 1996, pp. 295–352.
- [15] C.St.J.A. Nash-Williams, Hamiltonian circuits in graphs with vertices of large valency, in: L. Mirsky (Ed.), *Studies in Pure Mathematics*, Academic Press, London, 1971.
- [16] V. Rödl, A. Ruciński, Perfect matchings in ε -regular graphs and the Blow-up Lemma, *Combinatorica* 19 (1999) 437–452.
- [17] V. Rödl, A. Ruciński, A. Taraz, Hypergraph packing and graph embedding, *Combinatorics, Probab. Comput.* 8 (1999) 363–376.
- [18] A. Schrijver, Counting 1-factors in regular bipartite graphs, *J. Combin. Theory, Ser. B* 72 (1998) 122–135.
- [19] M. Simonovits, Vera T. Sós, Szemerédi’s partition and quasirandomness, *Random Struct. Algorithms* 2 (1991) 1–10.
- [20] E. Szemerédi, Regular partitions of graphs, *Colloques Internationaux CNRS No. 260—Problèmes Combinatoires et Théorie des Graphes*, Orsay, 1976, pp. 399–401.