Journal of Combinatorial Theory, Series B **78**, 115–122 (2000)
doi:10.1006/jctb.1999.1933, available online at http://www.idealibrary.com on IDEAL®

Vertex Partitions by Connected Monochromatic *k*-Regular Graphs

Gábor N. Sárközy and Stanley M. Selkow

Computer Science Department, Worcester Polytechnic Institute, Worcester, Massachusetts 01609

Communicated by the Managing Editors

Received October 28, 1997; revised May 11, 1999; accepted August 16, 1999

Generalizing a result of Erdős, Gyárfás and Pyber we show that there exists a constant c such that for any integers $r, k \geqslant 2$ and for any coloring of the edges of a complete graph with r colors, its vertices can be partitioned into at most $r^{c(r \log r + k)}$ connected monochromatic k-regular subgraphs and vertices. We also show that the same result holds for complete bipartite graphs, generalizing a result of Haxell.

1. INTRODUCTION

When A, B are disjoint subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B.

Definition 1. The bipartite graph $G = (A, B, E)$ is (ε, δ) super-regular if

$$X \subset A, \quad Y \subset B, \quad |X| > \varepsilon\,|A|, \quad |Y| > \varepsilon\,|B| \quad \text{imply} \quad e(X, Y) > \delta\,|X|\,|Y|$$

and furthermore,

$$\deg(a) \geqslant \delta\,|B| \qquad \text{for all} \quad a \in A,$$

$$\deg(b) \geqslant \delta\,|A| \qquad \text{for all} \quad b \in B.$$

We will often say simply that "the pair (A, B) is (ε, δ) super-regular" with the graph G implicit.

For any $r, k \geqslant 2$, let $f(r, k)$ denote the minimum number of connected monochromatic k-regular subgraphs and vertices which suffice to partition the vertices of any complete graph whose edges are r-colored. It is not obvious that $f(r, k)$ is a well-defined function. That is, it is not obvious that

0095-8956/00 $35.00

there is always a partition whose cardinality is independent of the order of the complete graph. Gyárfás in [2] conjectured the existence of $f(r, 2)$, and indeed Erdős, Gyárfás and Pyber in [1] proved that there exists a c such that $f(r, 2) \leqslant cr^2 \log r$. The generalization of this problem for k-regular graphs was initiated by Pyber, Rödl and Szemerédi in [8] who proved that in any r-coloring of the edges of the complete graph K_n there is a monochromatic k-regular subgraph for any $1 \leqslant k \leqslant c_r n$ where c_r is a constant depending on r. In our main theorem instead of just finding one monochromatic k-regular subgraph, we *partition* the vertex set into connected monochromatic k-regular subgraphs. This answers a question raised by Herman Servatius [10].

THEOREM 2. *There exists a constant c such that $f(r, k) \leqslant r^{c(r \log r + k)}$, i.e., for any $r, k \geqslant 2$ and for any coloring of the edges of a complete graph with r colors, its vertices can be partitioned into at most $r^{c(r \log r + k)}$ connected monochromatic k-regular subgraphs and vertices.*

The necessity of including isolated vertices in the partition follows from a coloring in which there is a vertex v and a color red such that an edge is red if and only if it is incident with v.

Erdős, Gyárfás and Pyber in [1] conjectured that $f(r, 2) = r$, where an edge and a vertex are degenerate cycles. Recently this conjecture was proved for $r = 2$ and $n \geqslant n_o$ by Łuczak, Rödl and Szemerédi [7].

Similarly as above we can define $f_b(r, k)$ for complete bipartite graphs $K_{n,n}$ instead of complete graphs. In [1] they also raised the question whether $f_b(r, 2)$ is also independent of n. This was proved recently by Haxell in [3] who showed that there exists a c such that $f_b(r, 2) \leqslant c(r \log r)^2$. Our second theorem generalizes this result as well.

THEOREM 3. *There exists a constant c such that $f_b(r, k) \leqslant r^{c(r \log r + k)}$, i.e., for any $r, k \geqslant 2$ and for any coloring of the edges of the complete bipartite graph $K_{n,n}$ with r colors, its vertices can be partitioned into at most $r^{c(r \log r + k)}$ connected monochromatic k-regular subgraphs and vertices.*

2. PROOF OF THEOREM 2

Let K be an r-colored copy of K_n. Generalizing the proofs in [1] and [3], we establish the bound on $f(r, k)$ in two steps.

- *Step* 1. We find a sufficiently large monochromatic super-regular pair (A_1, B_1) in K. After removing the pair (A_1, B_1) from K, we continue in this fashion. We greedily remove a number (which depends upon r and k) of super-regular pairs (A_i, B_i), $i \geqslant 2$ from the remainder in K and we find connected monochromatic k-regular spanning subgraphs in these super-regular pairs (A_i, B_i), $i \geqslant 2$.

- *Step* 2. We combine the remaining vertices with some vertices of (A_1, B_1) and we find a connected monochromatic k-regular spanning subgraph in the remainder of (A_1, B_1).

Here in this proof method, the greedy technique was introduced in [1]. The idea to make (A_1, B_1) super-regular comes from [3]. For our purposes we had to make (A_i, B_i) super-regular for every $i \geqslant 1$.

2.1. *Tools*

In this section we list our main tools. First we are going to use the following lemma of Komlós ([4], see also [3]).

LEMMA 4. *There exists a constant ε_0 such that if $\varepsilon \leqslant \varepsilon_0$, $t = (3/\varepsilon)\log(1/\varepsilon)$ and G_n is a graph with n vertices and cn^2 edges, then G_n contains an (ε, δ) super-regular subgraph (A_1, B_1) with*

$$|A_1| = |B_1| = m \geqslant (2c)^t \left\lfloor \frac{n}{2} \right\rfloor \qquad \textit{and} \qquad \delta \geqslant c.$$

We also use the following very special case of the Blow-up Lemma ([5], see also [3] and [9]).

LEMMA 5. *Given an $\varepsilon > 0$, if (A, B) is an (ε, δ) super-regular pair with $|A| = |B| = m \geqslant 1/\varepsilon$ and $\delta > 7\varepsilon$, then (A, B) is Hamiltonian.*

This lemma has the following consequence.

LEMMA 6. *Given an $\varepsilon > 0$ and an integer $k \geqslant 2$, if (A, B) is an (ε, δ) super-regular pair with $|A| = |B| = m \geqslant k/\varepsilon^2$ and $\delta > 9\varepsilon$, then (A, B) contains a connected k-regular spanning subgraph.*

Proof. Note that here it is not sufficient just to refer to the general Blow-up Lemma, since it gives the result only for sufficiently large m and for ε that is sufficiently small compared to δ. However, here we need these explicit estimations to get the numeric bound in Theorem 2. Thus instead, we remove the edges of the Hamiltonian cycle guaranteed by Lemma 5 and we apply the lemma again in the remainder, etc. We apply the lemma $\lfloor k/2 \rfloor$ times, and if k is odd once more to find a perfect matching. The conditions of Lemma 5 are always satisfied since after removing at most $\lfloor k/2 \rfloor$ Hamiltonian cycles, the pair (A, B) is still (ε, δ') super-regular with

$\delta' = \delta - 2\varepsilon > 7\varepsilon$. Indeed, for every $a \in A$ after the removals (and similarly for $b \in B$) we have

$$\deg(a) \geqslant \delta m - k = \left(\delta - \frac{k}{m}\right) m \geqslant (\delta - \varepsilon^2)\, m > \delta' m$$

Furthermore, if

$$X \subset A, \qquad Y \subset B, \quad |X| > \varepsilon m \geqslant \frac{k}{\varepsilon}, \quad |Y| > \varepsilon m \geqslant \frac{k}{\varepsilon},$$

then we have

$$e(X, Y) > \delta\, |X|\, |Y| - k(|X| + |Y|) = \left(\delta - \left(\frac{k}{|Y|} + \frac{k}{|X|}\right)\right) |X|\, |Y| >$$

$$> (\delta - 2\varepsilon)\, |X|\, |Y| = \delta'\, |X|\, |Y|.$$

2.2. *Step* 1

Let H_i be the subgraph of K with all edges of color i. Let i_1 be a color for which $e(H_{i_1}) \geqslant e(K)/r$. Let ε_0 be as in Lemma 4 and $\varepsilon = \varepsilon_0/50r$. Applying Lemma 4 to H_{i_1} there is a $\delta_1 \geqslant 1/4r$ and a pair (A_1, B_1) in color i_1 such that

- $|A_1| = |B_1| = m_1 \geqslant (1/2r)^t\, n$ where $t = (3/\varepsilon) \log(1/\varepsilon)$, and
- (A_1, B_1) is (ε, δ_1) super-regular.

Let $K_1 = K \backslash (A_1, B_1)$. Using Lemma 4 again in K_1, there is a color i_2, a $\delta_2 \geqslant 1/4r$ and a pair (A_2, B_2) in color i_2 such that

- $|A_2| = |B_2| = m_2 \geqslant (1/2r)^t\, (n - 2m_1)$, and
- (A_2, B_2) is (ε, δ_2) super-regular.

Removing (A_2, B_2) and continuing in this fashion always removing at least the fraction $2(1/2r)^t$ of the remaining vertices, after p steps the number of remaining vertices is at most

$$n\left(1 - 2\left(\frac{1}{2r}\right)^t\right)^p. \tag{1}$$

Defining

$$x = 2r^2(2er)^{\lceil k/2 \rceil} \qquad \text{and} \qquad x' = \max\left(\frac{m_1}{x^2}, \frac{(2r)^t\, k}{\varepsilon^2}\right), \tag{2}$$

we stop with the procedure when no more than x' vertices remain. Denote the last chosen super-regular pair by $(A_{p'}, B_{p'})$. Note that we may apply Lemma 6 for a pair (A_i, B_i), $1 \leqslant i \leqslant p'$, since $|A_i| = |B_i| \geqslant k/\varepsilon^2$.

In case $x' = (2r)^t k/\varepsilon^2$ (in other words we run out of room before completing our goal), we do not even need Step 2. The remaining vertices are going to be isolated vertices in the partitioning, and by using Lemma 6 in (A_i, B_i), $1 \leqslant i \leqslant p'$, the rest of K is partitioned by p' connected monochromatic k-regular graphs.

In the other case when $x' = m_1/x^2$ holds, we apply Lemma 6 only in (A_i, B_i), $2 \leqslant i \leqslant p'$, so K consists of (A_1, B_1), a set of $p' - 1$ connected monochromatic k-regular graphs, plus a set Y of fewer than m_1/x^2 vertices and we go to Step 2.

Note, that it follows from (1) that in either case we have

$$p' \leqslant \left\lceil \frac{(2r)^t}{2} (2 \log x + t \log(2r)) \right\rceil. \tag{3}$$

2.3. *Step* 2

We may make $|Y|$ even by removing an isolated vertex. We find an arbitrary partition $Y = Y' \cup Y''$ with $|Y'| = |Y''|$. The following theorem will help to combine the vertices in Y' with some vertices in B_1 and the vertices in Y'' with some vertices in A_1.

THEOREM 7. *If the edges of the complete bipartite graph (S, Y) are colored with r colors, $|S| = m$ and $|Y| < m/x^2$ (where x is given by (2)), then the vertices of Y can be covered by at most $rx(1 + \lceil k/2 \rceil) + 2r^2 \lceil k/2 \rceil$ vertex-disjoint connected monochromatic k-regular graphs and vertices.*

Proof. For each $y \in Y$ and $1 \leqslant i \leqslant r$, we define

$$N_i(y) = \{s \in S \mid (s, y) \text{ has color } i\},$$

and for $Y' \subset Y$ we define $N_i(Y') = \bigcap_{y \in Y'} N_i(y)$. Clearly Y can be partitioned into classes $Y_1, Y_2, \ldots, Y_r$ such that $|N_i(y)| \geqslant m/r$ for each $y \in Y_i$.

LEMMA 8. *For each Y_i, there is an a_i such that Y_i can be partitioned into classes $Y_{i0}, Y_{i1}, \ldots, Y_{ia_i}$ where*

- $|Y_{i0}| < 2r \lceil k/2 \rceil$,
- $|Y_{ij}| = \lceil k/2 \rceil$ *for* $1 \leqslant j \leqslant a_i$, *and*
- $|N_i(Y_{ij})| \geqslant rm/x$ *for* $1 \leqslant j \leqslant a_i$.

Proof. If $|Y_i| < 2r\lceil k/2\rceil$, the proof is trivial. Let H_i be the subgraph $S \times Y_i$ with all edges of color i. If $|Y_i| \geqslant 2r\lceil k/2\rceil$, then we have

$$\sum_{\substack{s\in S\\ \deg_{H_i}(s)\geqslant\lceil k/2\rceil}} \deg_{H_i}(s) \geqslant \frac{m}{r}|Y_i| - \left\lceil\frac{k}{2}\right\rceil m \geqslant \frac{m}{2r}|Y_i|.$$

We are going to count with multiplicity the number of subsets of Y_i of size $\lceil k/2\rceil$ with a common neighbor in S. Using Jensen's inequality,

$$\sum_{\substack{s\in S\\ \deg_{H_i}(s)\geqslant\lceil k/2\rceil}} \binom{\deg_{H_i}(s)}{\left\lceil\frac{k}{2}\right\rceil} \geqslant \frac{m}{2r}\binom{\frac{|Y_i|}{2r}}{\left\lceil\frac{k}{2}\right\rceil} \geqslant \frac{m}{2r}\left(\frac{|Y_i|}{2r\left\lceil\frac{k}{2}\right\rceil}\right)^{\lceil k/2\rceil}.$$

But there are only

$$\binom{|Y_i|}{\left\lceil\frac{k}{2}\right\rceil} \leqslant \left(\frac{e\,|Y_i|}{\lceil k/2\rceil}\right)^{\lceil k/2\rceil}$$

subsets of Y_i of size $\lceil k/2\rceil$. Thus there must be a $Y_{i1} \subset Y_i$ such that

$$|Y_{i1}| = \left\lceil\frac{k}{2}\right\rceil \qquad \text{and} \qquad |N_i(Y_{i1})| \geqslant \frac{m}{2r}\,\frac{\left(\frac{|Y_i|}{2r\lceil k/2\rceil}\right)^{\lceil k/2\rceil}}{\left(\frac{e\,|Y_i|}{\lceil k/2\rceil}\right)^{\lceil k/2\rceil}} = \frac{m}{2r(2er)^{\lceil k/2\rceil}} = \frac{rm}{x}.$$

Replacing Y_i by $Y_i \backslash Y_{i1}$ we repeat the procedure until for the leftover we have $|Y_{i0}| < 2r\lceil k/2\rceil$. We denote the number of repetitions by a_i. This completes the proof of Lemma 8.

For each Y_i we define an auxiliary graph G_i with vertices $\{Y_{i1}, Y_{i2}, ..., Y_{ia_i}\}$ and edges

$$\left\{(Y_{ij}, Y_{il}) \,\middle|\, |N_i(Y_{ij}) \cap N_i(Y_{il})| \geqslant \frac{m}{x^2} > |Y|\right\}.$$

LEMMA 9. *The size of a maximum independent set of G_i is less than x.*

Proof. Assume indirectly that $\{w_1, w_2, ..., w_x\} \subset \{Y_{i1}, Y_{i2}, ..., Y_{ia_i}\}$ is an independent set of vertices of G_i. If $w_j = Y_{ij}$, then we define $N_i(w_j) = N_i(Y_{ij})$. Hence we have $|N_i(w_j)| \geqslant rm/x$ for $1 \leqslant j \leqslant x$. But then

$$m \geqslant \left| \bigcup_{1 \leqslant j \leqslant x} N_i(w_j) \right| \geqslant rm - \sum_{1 \leqslant j < l \leqslant x} |N_i(w_j) \cap N_i(w_l)| \geqslant$$
$$\geqslant rm - \frac{x^2}{2} \frac{m}{x^2} = \left(r - \frac{1}{2} \right) m > m.$$

By contradiction, G_i can not have an independent set of x vertices, finishing the proof of Lemma 9.

By Lemma 9 and a theorem of Pósa [6], the vertices of G_i can be partitioned into at most x cycles (and edges and vertices), and thus the vertices of $\bigcup_{1 \leqslant i \leqslant r} G_i$ can be partitioned into at most rx cycles (and edges and vertices). The vertices in this partitioning will be isolated vertices ($\lceil k/2 \rceil$ vertices of Y for each) and the edges are considered cycles of length 2. Between every adjacent pair of vertices on these cycles, we insert disjoint sets from S. Between adjacent vertices Y_{ij} and Y_{il}, we insert $S_{ij} \subset S$ such that $|S_{ij}| = \lceil k/2 \rceil$ and $S_{ij} \times (Y_{ij} \cup Y_{il})$ is monochromatic in color i. Inserting these sets (from S) between the corresponding pairs of sets (from Y) on a cycle yields a new "cycle", $Z_1, Z_2, ..., Z_{2p}$ of sets of vertices of size $\lceil k/2 \rceil$. The graph with vertices $\bigcup_{1 \leqslant j \leqslant 2p} Z_j$ and edges $\bigcup_{1 \leqslant j < 2p} (Z_j \times Z_{j+1}) \cup (Z_1 \times Z_{2p})$ is a connected monochromatic $k + (k \bmod 2)$-regular graph. For odd k, removing a perfect matching in each of $Z_{2j+1} \times Z_{2j+2}$ for $0 \leqslant j < p$ yields a connected monochromatic k-regular graph. Hence the vertices of $S \times Y$ can be partitioned into at most rx connected monochromatic k-regular graphs plus at most $rx \lceil k/2 \rceil + 2r^2 \lceil k/2 \rceil$ vertices. This finishes the proof of Theorem 7.

Applying Theorem 7 for $S = B_1$ and Y', we obtain a set of at most $rx(1 + \lceil k/2 \rceil) + 2r^2 \lceil k/2 \rceil$ connected monochromatic k-regular graphs and vertices that partition the vertices in Y' and a subset of B' of B_1. Similarly we have a set of at most $rx(1 + \lceil k/2 \rceil) + 2r^2 \lceil k/2 \rceil$ connected monochromatic k-regular graphs and vertices that partition the vertices in Y'' and a subset of A' of A_1. Assuming $|A'| < |B'|$, we add $|B'| - |A'|$ additional isolated vertices from A_1 to A', thus now $|A_1 \backslash A'| = |B_1 \backslash B'|$. Finally we apply Lemma 6 for $H_{i_1}|_{(A_1 \backslash A') \cup (B_1 \backslash B')}$. It is not hard to check that the conditions of Lemma 6 are still satisfied.

Thus, using (2) and (3), in all cases altogether in our covering the number of connected monochromatic k-regular graphs and vertices we used is at most

$$p' + 3 \left(rx \left(1 + \left\lceil \frac{k}{2} \right\rceil \right) + 2r^2 \left\lceil \frac{k}{2} \right\rceil \right) + \frac{(2r)^t k}{\varepsilon^2} + 1 \leqslant r^{c(r \log r + k)}$$

with some constant c.

The proof of Theorem 3 is almost identical and is omitted.

3. CONCLUDING REMARKS

The obvious open problem is to determine $f(r, k)$ and $f_b(r, k)$. As we mentioned above Erdős, Gyárfás and Pyber in [1] conjectured that $f(r, 2) = r$, where an edge and a vertex are degenerate cycles. With a somewhat more tedious calculation we can get $c = 200$ in Theorems 2 and 3, however, since we think that it is still far from optimal, we omit the details. Furthermore, it would be an interesting problem to find other families of graphs for which the partition number is independent of n.

4. ACKNOWLEDGEMENT

We thank one of the referees for helpful remarks.

REFERENCES

1. P. Erdős, A. Gyárfás, and L. Pyber, Vertex coverings by monochromatic cycles and trees, *J. Combin. Theory B* **51** (1991), 90–95.
2. A. Gyárfás, Covering complete graphs by monochromatic paths, *in* "Irregularities of Partitions," Algorithms and Combinatorics, Vol. 8, pp. 89–91, Springer-Verlag, 1989.
3. P. Haxell, Partitioning complete bipartite graphs by monochromatic cycles, *J. Combin. Theory B* **69** (1997), 210–218.
4. J. Komlós and M. Simonovits, Szemerédi's Regularity Lemma and its applications in graph theory, *in* "Combinatorics, Paul Erdős is Eighty" (D. Miklós, V. T. Sós, and T. Szőnyi, Eds.), Bolyai Society Mathematical Studies, Vol. 2, pp. 295–352, János Bolyai Mathematical Society, Budapest, 1996.
5. J. Komlós, G. N. Sárközy, and E. Szemerédi, Blow-up Lemma, *Combinatorica* **17** (1) (1997), 109–123.
6. L. Lovász, "Combinatorial Problems and Exercices," p. 56, North–Holland, Amsterdam, 1979.
7. T. Łuczak, V. Rödl, and E. Szemerédi, Partitioning two-colored complete graphs into two monochromatic cycles, *Probability, Combinatorics and Computing* **7** (1998), 423–436.
8. L. Pyber, V. Rödl, and E. Szemerédi, Dense graphs without 3-regular subgraphs, *J. Combin. Theory B* **63** (1995), 41–54.
9. G. N. Sárközy, Fast parallel algorithms for finding Hamiltonian cycles and trees in graphs, Technical Report 93–81, DIMACS, Rutgers University.
10. H. Servatius, private communication.