On *k*-Ordered Hamiltonian Graphs

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Abstract: A Hamiltonian graph G of order n is k-ordered, $2 \leq k \leq n$, if for every sequence $v_1, v_2, \ldots, v_k$ of k distinct vertices of G, there exists a Hamiltonian cycle that encounters $v_1, v_2, \ldots, v_k$ in this order. Define $f(k, n)$ as the smallest integer m for which any graph on n vertices with minimum degree at least m is a k-ordered Hamiltonian graph. In this article, answering a question of Ng and Schultz, we determine $f(k, n)$ if n is sufficiently large in terms of k. Let $g(k, n) = \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor - 1$. More precisely, we show that $f(k, n) = g(k, n)$ if $n \geq 11k - 3$. Furthermore, we show that $f(k, n) \geq g(k, n)$ for any $n \geq 2k$. Finally we show that $f(k, n) > g(k, n)$ if $2k \leq n \leq 3k - 6$. © 1999 John Wiley & Sons, Inc. J Graph Theory 32: 17–25, 1999

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1. INTRODUCTION

A. Notations and Definitions

For basic graph concepts, see the monograph of Bollobás [1]. $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. $N(v)$ is the set of neighbors of $v \in V$. Hence, the size of $N(v)$ is $|N(v)| = \deg(v) = \deg_G(v)$, the degree of v. $\delta(G)$ stands for the minimum degree in G. For a vertex $v \in V$ and set $U \subset V - \{v\}$, we write $\deg(v, U)$ for the number of edges from v to U. For a graph G and a subset U of its vertices, $G|_U$ is the restriction to U of G (or the subgraph of G induced by the vertices of U). Let $[k] = \{1, 2, \ldots, k\}$.

B. k-Ordered Hamiltonian Graphs

Let G be a graph on $n \geq 3$ vertices. A *Hamiltonian cycle* (*path*) of G is a cycle (path) containing every vertex of G. A *Hamiltonian graph* is a graph containing a Hamiltonian cycle. A classical result of Dirac [2] asserts that if $\delta(G) \geq n/2$, then G is Hamiltonian. A *Hamiltonian-connected* graph is a graph in which every pair of vertices can be connected with a Hamiltonian path. Note that, by another classical result (see [1]), if $\delta(G) \geq (n+1)/2$, then G is Hamiltonian-connected.

The following interesting concept was created by Chartrand: For a positive integer $2 \leq k \leq n$, and for a sequence $S = v_1, v_2, \ldots, v_k$ of k distinct vertices, a cycle C in G is called a $v_1 - v_2 - \cdots - v_k$*-cycle*, or simply an S*-cycle*, if the vertices of S are encountered on C in the specified order. For a Hamiltonian graph G, we say that G is k*-ordered* if, for every sequence $S = v_1, v_2, \ldots, v_k$ of k distinct vertices, there exists a Hamiltonian S-cycle. It is not hard to see that every Hamiltonian graph is both 2-ordered and 3-ordered. Furthermore, a Hamiltonian graph G of order n is n-ordered if and only if $G = K_n$. Also, if G is k-ordered, then G is l-ordered for every $2 \leq l \leq k$ (see [5]).

A natural question is whether we can obtain a Dirac-type condition on the minimum degree for guaranteeing that the graph is a k-ordered Hamiltonian graph. Indeed, the first result of this type was obtained in [5]. In this article, it was shown (among other results) that, if $3 \leq k \leq n$ and $\delta(G) \geq \frac{n}{2} + k - 3$, then G is a k-ordered Hamiltonian graph. The authors raised the question of whether this can be improved. In this article, our main goal is to determine the best possible bound under the restriction that n is sufficiently large in terms of k. Define $f(k, n)$ as the smallest integer m for which any graph on n vertices with minimum degree at least m is a k-ordered Hamiltonian graph. Let

$$g(k, n) = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1.$$

We show that $f(k, n) = g(k, n)$, if $n \geq 11k - 3$. Furthermore, we show that $g(k, n)$ is always a lower bound for any $n \geq 2k$. Finally, somewhat surprisingly, we show that sometimes $f(k, n) > g(k, n)$. More precisely, we have the following.

Theorem 1. *For positive integers k, n with $n \geq 2k$ we have*

(a) $f(k,n) = g(k,n)$ *for* $n \geq 11k - 3$,
(b) $f(k,n) \geq g(k,n)$ *for any* $n \geq 2k$,
(c) $f(k,n) > g(k,n)$ *for* $2k \leq n \leq 3k - 6$.

The rest of the article is organized as follows. We prove (a) in Section 2. In view of (b), it is enough to show that, if G is a graph on n vertices with $\delta(G) \geq g(k,n)$ where $n \geq 11k - 3$, then G is a k-ordered Hamiltonian graph. Let $S = v_1, v_2, \ldots, v_k$ be any sequence drawn from $V(G)$. First, in Section 2.A we show that an S-cycle exists and then in Section 2.B we show that a maximum S-cycle is Hamiltonian. We give the simple proofs of (b) and (c) in Sections 3 and 4. We finish with some remarks and open problems in Section 5.

2. PROOF OF (a)

Let G be a graph on n vertices with

$$\delta(G) \geq g(k,n) = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1, \tag{1}$$

where

$$n \geq 11k - 3. \tag{2}$$

We have to show that G is a k-ordered Hamiltonian graph. We may assume that $k > 4$, since otherwise this is trivial. Let $S = v_1, v_2, \ldots, v_k$ be any sequence drawn from $V(G)$. First, we construct an S-cycle.

A. Construction of an S-Cycle

Call $C = (v_1, P_1, v_2, P_2, \ldots, v_k, P_k)$ a partial S-cycle, if each P_i is either empty or a $v_i - v_{i+1}$ path with at most three internal vertices, and the internal vertices of the P_i are pairwise disjoint and disjoint from S. So a partial S-cycle is an S-cycle if all the P_i are nonempty. A partial S-cycle C is optimal, if as many as possible of the P_i are nonempty and subject to this C has as few vertices as possible. Suppose for a contradiction that C is an optimal partial S-cycle, but C is not an S-cycle. Say P_i is empty and set $x = v_i$ and $y = v_{i+1}$. Let X be the vertex set of C. Let $A = N(x) \backslash X$, $A' = (N(A) \cup A) \backslash X$, $B = N(y) \backslash X$, and $B' = (N(B) \cup B) \backslash X$.

For all nonadjacent $u, v \in V(G)$, we have from (1)

$$|N(u) \cap N(v)| \geq 2\left(\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1\right) - (n-2) = 2\left\lceil \frac{n}{2} \right\rceil + 2\left\lfloor \frac{k}{2} \right\rfloor - n. \tag{3}$$

This last expression is always at least $k - 1$, and, unless k is odd, it is at least k.

Note that from (3) we get $(N(x) \cap N(y)) \backslash S \neq \emptyset$ and so, by the optimality of C, $|P_h \backslash \{v_h, v_{h+1}\}| \leq 1$, for some h. Also $P_i = \emptyset$. So

$$|X| = k + \sum_{j=1}^{k} |P_j \backslash \{v_j, v_{j+1}\}| \leq k + 3(k-2) + 1 + 0 \leq 4k - 5. \tag{4}$$

Using (1), we get

$$|A|, |B| \geq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1 - |X| \geq \frac{n+k-3}{2} - |X|. \tag{5}$$

By the optimality of C, A' and B' are disjoint, no vertex in $V(G) \backslash C$ is adjacent to four vertices in $P_j \backslash \{v_{j+1}\}$, for any j, and no vertex in $V(G) \backslash C$ is adjacent to both x and y. Thus, for any vertex $v \in V(G) \backslash X$,

$$\text{deg}(v, X) \leq 3k - 3. \tag{6}$$

Let $A'' = \{v \in X || N(v) \cap A'| > 1\}$ and $B'' = \{v \in X || N(v) \cap B'| > 1\}$. Then $X = A'' \cup B''$.

For all nonadjacent $s \in A', t \in A' \cup (X \backslash B'')$ (and similarly for $s \in B'$ and $t \in B' \cup (X \backslash A''))$ using (1), (2) (and actually this is the only point where we need exactly this bound), (4), (5), and (6), we have

$$|(N(s) \cap N(t)) \backslash X| \geq 3. \tag{7}$$

Indeed, since $N(s) \cap (B \cup \{s,t\}) = \emptyset$ and $|N(t) \cap (B \cup \{s,t\})| \leq 1$,

$$\begin{aligned}
&|(N(s) \cap N(t)) \backslash X| \\
&\quad \geq \text{deg}(s) + \text{deg}(t) - 1 - |V(G) \backslash (B \cup \{s,t\})| - |(N(s) \cap N(t)) \cap X| \\
&\quad \geq (n + k - 4) - \left(n - \left(\frac{n+k+1}{2} - |X|\right)\right) - \text{deg}(s, X) \\
&\quad \geq \frac{n + 3k - 7}{2} - (4k - 5) - (3k - 3) \\
&\quad \geq 3.
\end{aligned}$$

We shall obtain a contradiction by finding a better partial S-cycle. We distinguish two cases.

Case 1. There exist $a \in A, b \in B$, and $j \in [k]$ such that both $N(a) \cap N(b) \cap (P_j \backslash \{v_j, v_{j+1}\}) \neq \emptyset$ and either $\{v_j, v_{j+1}\} \subset A''$ or $\{v_j, v_{j+1}\} \subset B''$. Say $\{v_j, v_{j+1}\} \subset A''$ and $r \in N(a) \cap N(b) \cap (P_j \backslash \{v_j, v_{j+1}\})$. Let $P_i' = (x, a, r, b, y)$. Since $v_j, v_{j+1} \in A''$, there exist $s \in (A' \cap N(v_j)) \backslash \{a\}$ and $t \in (A' \cap N(v_{j+1})) \backslash \{a\}$. If s is adjacent to t, then let $P_j' = (v_j, s, t, v_{j+1})$. Otherwise, by (7), there exists $q \in (N(s) \cap N(t)) \backslash (X \cup \{a, b\})$. Let $P_j' = (v_j, s, q, t, v_{j+1})$. In either case, P_j' is a path disjoint from P_i'. For $h \in [k] \backslash \{i, j\}$, let $P_h' = P_h$. Then $(v_1, P_1', v_2, P_2', \ldots, v_k, P_k')$ contradicts the assumption that C is optimal.

Case 2. Not Case 1. For an even k, we define a partition P of $C\backslash\{v_i, v_{i+1}\}$ as

$$P = \{P_{i+1}\backslash\{v_{i+1}\}, P_{i+2}\backslash\{v_{i+2}\}, \dots, P_{i-1}\backslash\{v_{i-1}, v_i\}\}.$$

The sets in this partition are denoted by Q_j, where $Q_j \subset P_j$. For an odd k, since $x \in A''$ and $y \in B''$, there exists $h \in [k]$ such that $\{v_h, v_{h+1}\} \subset A''$ or $\{v_h, v_{h+1}\} \subset B''$. Then, by the case, $N(a) \cap N(b) \cap (P_h\backslash\{v_h, v_{h+1}\}) = \emptyset$. Without loss of generality, $h < i$. Here we define the partition P of $C\backslash(\{v_i, v_{i+1}\} \cup (P_h\backslash\{v_h, v_{h+1}\}))$ as

$$P = \{P_{h+1}\backslash\{v_{h+2}\}, \dots, P_{i-1}\backslash\{v_i\}, P_{i+1}\backslash\{v_{i+1}\}, \dots, P_{h-1}\backslash\{v_{h-1}\}\}.$$

Again the sets in this partition are denoted by Q_j.

The pigeon hole principle and (3) imply for both even and odd k-s that, for every $a \in A, b \in B$, there exists $j = j(a, b)$ such that

$$|N(a) \cap N(b) \cap Q_j| \geq 2.$$

Since from (2), (4), and (5) we have $|A|, |B| \geq k$, there exist distinct $a_1, a_2 \in A$, distinct $b_1, b_2 \in B$, and $j \in [k]$ such that $j(a_1, b_1) = j(a_2, b_2) = j$. Let $u = v_j$ and $v = v_{j+1}$. By the case, we may assume that $u \in A''\backslash B''$ and $v \in B''\backslash A''$. This implies that for one of the pairs we have at least 2 common neighbors, which are internal vertices of P_j. Then there exist distinct $s_1, s_2 \in P_j\backslash\{u, v\}$ such that (a_1, s_1, b_1) and (a_2, s_2, b_2) are paths. Assume that P_i starts with (x, s_2, s_1). (Otherwise P_i ends with (s_1, s_2, v).) Let $P_i' = (x, a_1, s_1, b_1, y)$. Then P_i' is a path. If b_2 is adjacent to v, then let $P_j' = (u, s_2, b_2, v)$. Otherwise by (7) there exists $r \in N(b_2) \cap N(v)\backslash(X \cup \{a_1, b_1, b_2\})$. Let $P_j' = (u, s_2, b_2, r, v)$. In either case, P_j' is a path disjoint from P_i'. Let $P_h' = P_h$ for all $h \in [k]\backslash\{i, j\}$. Then $(v_1, P_1', v_2, P_2', \dots, v_k, P_k')$ is a contradiction to the optimality of C.

B. Maximum S-Cycle is Hamiltonian

Let C be a maximum S-cycle. If C is Hamiltonian, then we are done. Otherwise, let $H = V(G)\backslash C$. Let $c = |C|$ and $h = |H|$. Then no vertex w of H is adjacent to two consecutive vertices $y, y' \in C$, since otherwise we could insert w between y and y'. Thus,

$$\delta(G|_H) \geq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1 - \left\lfloor \frac{c}{2} \right\rfloor \geq \left\lceil \frac{h}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1. \tag{8}$$

So $G|_H$ is Hamiltonian-connected and $h \geq k$.

Let $N = \{y \in C | y \text{ is adjacent to a vertex in } H\}$. No two vertices $y, y' \in N$ are consecutive in C, since $G|_H$ is Hamiltonian-connected and C is maximum. So

$$0 = \min\{\deg(y, H) | y \in C\} \geq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1 - (c - 1),$$

and, thus,

$$c \geq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor \quad \text{and} \quad h \leq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor. \tag{9}$$

Furthermore,

$$\min\{\deg(w, C) | w \in H\} \geq \delta(G) - (h-1). \tag{10}$$

Thus, from the above, we have

$$k \leq h \leq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor.$$

Again let P_i be the $v_i - v_{i+1}$ path on C whose intersection with S has size 2. We show first that we cannot have an $i \in [k]$ and vertices y, y', w, w' such that

$$y, y' \in P_i, w, w' \in H, y \neq y', w \neq w', (y, w), (y', w') \in E(G). \tag{11}$$

Assume indirectly that there are y, y', w, w' satisfying (11), and subject to this, choose y and y' as close as possible. Let w, P, w' be a Hamiltonian path in $G|_H$. By the choice of y and y', w' is not adjacent to any vertex on the path Q strictly between y and y' on C. Let $q = |Q|$. Thus,

$$\frac{n}{2} + \left\lfloor \frac{k}{2} \right\rfloor - 1 \leq \deg(w') \leq (h-1) + \frac{c-q+1}{2},$$

$$n + 2\left\lfloor \frac{k}{2} \right\rfloor \leq 2h + c - q + 1,$$

$$q + 2\left\lfloor \frac{k}{2} \right\rfloor \leq h + 1,$$

$$q < h.$$

But then $(C - Q) + (w, P, w')$ is a longer S-cycle than C, a contradiction. Hence, we may assume that we do not have y, y', w, w' satisfying (11).

Based on the size of h, we distinguish the following cases.

Case 1. $k \leq h \leq \lfloor \frac{n}{2} \rfloor - \lfloor \frac{k}{2} \rfloor - 1$. In this case, (1) and (10) imply that we have

$$\min\{\deg(w, C) | w \in H\} > k.$$

Then for each $w \in H$, there exist $i_w \in [k]$ and distinct $y < y' \in N(w) \cap C$ such that $v_{i_w}, \ldots, y, Q_w, y', \ldots, v_{i_w+1}$ is a path in C. If $k < h$, there exist $i \in [k]$ and distinct $w, w' \in H$ such that $i_w = i = i_{w'}$. It follows that there exist vertices y, y', w, w' satisfying (11), a contradiction. Thus, we may assume that $h = k$ and, furthermore, for all $i \in [k]$ there exists a $w_i \in H$ such that $N_{G|_C}(w_i) \subset P_i$. However, in this case from (2) and (10), we get

$$|C| \geq k(\delta(G) - k) \geq k\frac{n-k-3}{2} > n,$$

a contradiction.

Case 2. $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{k}{2} \rfloor - 1 \le h \le \lfloor \frac{n}{2} \rfloor - \lfloor \frac{k}{2} \rfloor$.

We have

$$\min\{\deg(w, C) | w \in H\} \ge k - 1. \tag{12}$$

Furthermore, from (9) for every vertex $y \in C \backslash N$, we have

$$\deg(y, C) = \deg(y) \ge \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1 \ge c - 2. \tag{13}$$

Thus, for each vertex $y \in C \backslash N$ we have at most one vertex in C that is not adjacent to y, and each vertex in H has at least $k - 1$ neighbors in C. Again we may assume that we have no vertices y, y', w, w' satisfying (11). However, then there exist $i \in [k], y \in P_i \backslash \{v_i, v_{i+1}\}, y' \in P_{i+1} \backslash \{v_{i+1}, v_{i+2}\}, y'' \in P_{i+2} \backslash \{v_{i+2}, v_{i+3}\}, w, w', w'' \in H$ such that $(y, w), (y', w'), (y'', w'') \in E(G)$. Let z, z', z'' precede y, y', y'' on C. We may assume that z' is adjacent to z (otherwise z' is adjacent to z'') and let P be a Hamiltonian path in $G|_H$ connecting w and w'. Then we get a Hamiltonian S-cycle by

$$z, z', \text{ part of } C \text{ from } z' \text{ back to } y, w, P, w', y', \text{ rest of } C,$$

a contradiction. This finishes Case 2 and the proof of (a).

3. PROOF OF (b)

Let $n \ge 2k$. We consider the graph G with vertices

$$\{u_1, \ldots, u_{\lfloor \frac{n}{2} \rfloor}, w_1, \ldots, w_{\lceil \frac{n}{2} \rceil}\}$$

such that $U = \{u_1, \ldots, u_{\lfloor \frac{n}{2} \rfloor}\}$ and $W = \{w_1, \ldots, w_{\lceil \frac{n}{2} \rceil}\}$ induce complete sub graphs of G. The edges of G between U and W are

$$(U \times \{w_1, \ldots, w_{\lfloor \frac{k}{2} \rfloor}\}) \cup (W \times \{u_1, \ldots, u_{\lfloor \frac{k}{2} \rfloor - 1}\}).$$

It is easily seen that

$$\delta(G) = g(k, n) - 1 = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 2,$$

as required. Furthermore, G does not contain a Hamiltonian cycle which encounters

$$u_{\lfloor \frac{k}{2} \rfloor} - w_{\lfloor \frac{k}{2} \rfloor + 1} - u_{\lfloor \frac{k}{2} \rfloor + 1} - w_{\lfloor \frac{k}{2} \rfloor + 2} - \cdots$$
$$- u_{2\lfloor \frac{k}{2} \rfloor - 1} - w_{2\lfloor \frac{k}{2} \rfloor} (-u_{2\lfloor \frac{k}{2} \rfloor} \text{ if } k \text{ is odd})$$

in this order. This follows from the fact that every $u_i - w_{i+1}$ and $w_i - u_i$ transition uses at least one vertex from the vertices

$$\{w_1, \ldots, w_{\lfloor \frac{k}{2} \rfloor}\} \cup \{u_1, \ldots, u_{\lfloor \frac{k}{2} \rfloor - 1}\}.$$

However, the number of transitions is always more than the number of vertices in this set. Here we also used the fact that we have enough vertices in U and W, since $n \geq 2k$. Thus, G is not a k-ordered Hamiltonian graph, finishing the proof of (b).

4. PROOF OF (c)

Let

$$2k \leq n \leq 3k - 6. \tag{14}$$

Here let G consist of four parts. First, the k special vertices $\{v_1, v_2, \ldots, v_k\}$ are all adjacent to each other, except that v_i is not adjacent to v_{i-1} and v_{i+1} (for $i = 1$, we put $v_{i-1} = v_k$ and for $i = k, v_{i+1} = v_1$). There are four (or five if $n \not\equiv k$ mod 2) vertices $\{y_1, y_2, y_3, y_4(, y_5)\}$ that are adjacent to all other vertices, including each other. There are two sets V_1 and V_2 such that

$$|V_1| = |V_2| = \frac{n-k-4}{2} \text{ if } n \equiv k \; mod \; 2,$$

and

$$|V_1| = |V_2| = \frac{n-k-5}{2} \text{ if } n \not\equiv k \; mod \; 2.$$

Furthermore, the vertices in $V_1 \cup V_2$ are adjacent to all other vertices with the exception that v_i for an odd i is not adjacent to any vertex in V_2, and v_i for an even i is not adjacent to any vertex in V_1.

We have

$$\deg(v_i) = k - 3 + 4 + \frac{n-k-4}{2} = \frac{n}{2} + \frac{k}{2} - 1 = g(k,n) \quad \text{if} \quad n \equiv k \; mod \; 2,$$

$$\deg(v_i) = k - 3 + 5 + \frac{n-k-5}{2} = \frac{n}{2} + \frac{k}{2} - \frac{1}{2} \geq g(k,n) \quad \text{if} \quad n \not\equiv k \; mod \; 2,$$

and, from (14),

$$\deg(z) \geq n - \left\lceil \frac{k}{2} \right\rceil \geq g(k,n) \quad \text{for all} \quad z \in V_1 \cup V_2.$$

Let the sequence S be $v_1, v_2, \ldots, v_k$. Clearly, there is no S-Hamiltonian cycle, since

$$\frac{n-k-4}{2} < k - 4 \text{ and } \frac{n-k-5}{2} < k - 5$$

(here we used (14) again). This finishes the proof of (c).

5. REMARKS AND OPEN PROBLEMS

In an earlier version of this article, we used the Regularity Lemma, Blow-up Lemma method (see, e.g., [3] and [4]) to obtain (a) for $n \geq ck$, where the constant c is very large. Then, as it happens in many applications of the Regularity Lemma, we found the more exact approach of this article to yield a much better constant. However, one advantage of the Regularity Lemma approach is that it gives a pancyclicity-type result as well, namely we can find an S-cycle of length s for any $4k \leq s \leq n$. Furthermore, it has a fast parallel algorithmic implementation as well.

The obvious open problem is to determine $f(k, n)$ for *every* $2 \leq k \leq n$.

Another open problem is to determine the best possible Ore-type condition. In [5], it is shown that, if $3 \leq k \leq n$ and deg$(u) +$ deg$(v) \geq n + 2k - 6$ for every pair u, v of nonadjacent vertices of G, then G is a k-ordered Hamiltonian graph.

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