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Note

Counting irregular multigraphs

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Abstract

Gagliardi et al. (1996, unpublished manuscript) defined an irregular multigraph to be a loopless multigraph with degree sequence $n, n-1, \ldots, 1$, and they posed the problem of determining the number of different irregular multigraphs f_n on n vertices. In Gagliardi et al. (1996) they showed that if $n \equiv 0$ or $3 \pmod 4$ then $f_n > n-1$. In this note our aim is to show that there are constants $1 < c_1 < c_2$ and $n_0 > 0$ such that if $n \geqslant n_0$ and $n \equiv 0$ or $3 \pmod 4$ then $(c_1)^{n^2} < f_n < (c_2)^{n^2}$. Indeed, we show that $c_1 = 1.19$ and $c_2 = 1.65$ can be chosen. © 1999 Elsevier Science B.V. All rights reserved

In this note we consider loopless multigraphs. $V(G)$ denotes the vertex set, $E(G)$ denotes the edge set of the multigraph G. For two multigraphs G and H, the union of G and H, written $G \cup H$, has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Gagliardi et al. [1,2] defined an *irregular* multigraph to be a loopless multigraph with degree sequence $n, n-1, \ldots, 1$, and they posed the problem of determining the number of different irregular multigraphs on n vertices. We define f_n to be the number of irregular multigraphs on n vertices.

As Gagliardi et al. [1] show, the even parity of $\sum_{1 \leqslant i \leqslant n} d_i$ clearly implies that $n \equiv 0$ or $3 \pmod 4$ is a necessary condition for $f_n > 0$. They also established that if $n \equiv 0$ or $3 \pmod 4$, then $f_n > n-1$. Our goal is to provide the following bounds for f_n. Note that

$$f_n = [z_n^n, \ldots, z_1] \prod_{i \neq j} (1 + z_i z_j)^n = [z_n^n, \ldots, z_1] \prod_{i \neq j} \frac{1}{1 - z_i z_j}.$$

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Theorem 1. *There are constants* $1 < c_1 < c_2$ *and* $n_0 > 0$ *such that if* $n \geqslant n_0$ *and* $n \equiv 0$ *or* $3 \pmod 4$ *then* $(c_1)^{n^2} < f_n < (c_2)^{n^2}$. *Indeed, we show that* $c_1 = 1.19$ *and* $c_2 = 1.65$ *can be chosen.*

Proof. For the lower bound, we partition the n vertices into sets A and B of sizes d and $n-d$ respectively, with $d = d(n) = \lambda(n-1)/(\lambda+1)$, where λ is a constant to be chosen later with the property that d is divisible by 4. We place fixed irregular multigraphs on A and B (these exist since $4|d$ and $n-d \equiv 0$ or $3 \pmod 4$). If we superimpose on B *any* d-regular multigraph B^*, then $A \cup B \cup B^*$ is an irregular multigraph on n vertices. Since any d-regular multigraph B^* superimposed on B yields a unique irregular multigraph $A \cup B \cup B^*$, the number of d-regular (*labeled*) *graphs* on $n-d$ vertices is a lower bound on f_n. A deep result of McKay and Wormald [3] implies that given the above conditions on d, as $n-d \to \infty$ the number of (labeled) d-regular graphs is at least

$$\frac{C}{(2\pi(n-d)\lambda^{d+1}(1-\lambda)^{n-2d})^{(n-d)/2}},$$

where C is an absolute constant. Thus, we get

$$\begin{aligned} f_n &\geqslant \frac{1}{(\lambda^{d+1}(1-\lambda)^{n-2d})^{(n-d)/2}} \frac{C}{(2\pi(n-d))^{(n-d)/2}} \\ &\geqslant (\lambda^{-\lambda/2(\lambda+1)^2}(1-\lambda)^{(\lambda-1)/2(\lambda+1)^2})^{n^2} \frac{C}{(2\pi(n-d))^{(n-d)/2}}. \end{aligned}$$

Choosing λ to be the largest real number that is at most 0.3 and for which d is divisible by 4, we get that if n is sufficiently large, then $f_n > (1.19)^{n^2}$.

To establish the upper bound, we note that if for some $1 \leqslant k \leqslant n-1$, v_k has j neighbors among $\{v_{k+1}, \ldots, v_n\}$ (where $0 \leqslant j \leqslant k$), then the number of ways to distribute these j edges among $\{v_{k+1}, \ldots, v_n\}$ is bounded from above by the number of ways to sample, with replacement, j elements from $n-k$ elements, or

$$\binom{n-k+j-1}{j}.$$

Thus an upper bound on the number of ways to distribute the edges is

$$\begin{aligned} f_n &\leqslant \prod_{1 \leqslant k \leqslant n-1} \left(\sum_{0 \leqslant j \leqslant k} \binom{n-k+j-1}{j} \right) = \prod_{1 \leqslant k \leqslant n-1} \binom{n}{k} = \prod_{1 \leqslant k \leqslant n} \binom{n}{k} \\ &\leqslant \prod_{1 \leqslant k \leqslant n} \frac{n^n}{k^k (n-k)^{n-k}} = \exp\left(\log\left(\prod_{1 \leqslant k \leqslant n} \frac{n^n}{k^k (n-k)^{n-k}} \right) \right) \\ &= \exp\left(\sum_{1 \leqslant k \leqslant n} \log\left(\frac{n^n}{k^k (n-k)^{n-k}} \right) \right) \\ &= \exp\left(n^2 \log n - \sum_{1 \leqslant k \leqslant n} k \log k - \sum_{1 \leqslant k \leqslant n} (n-k) \log(n-k) \right) \end{aligned}$$

$$\leqslant \exp\left(n^2\log n - 2\int_0^n x\log x\,\mathrm{d}x\right) = \exp\left(n^2\log n - 2\left[\frac{x^2\log x}{2} - \frac{x^2}{4}\right]_0^n\right)$$

$$= \exp\left(\frac{n^2}{2}\right) = (\sqrt{\mathrm{e}})^{n^2},$$

where $\sqrt{\mathrm{e}} \approx 1.64872$. □

We note, that for the lower bound of the previous theorem, we can allow A to be any irregular multigraph and B to be a set of irregular multigraphs such that if we superimpose any d-regular graph on distinct members of B it is impossible to generate an irregular multigraph in two different ways. Using this idea, we can raise the value of c_1 to be roughly 1.3. We omit the details, since the computation is somewhat tedious and the result is still far from the upper bound. The determination of the asymptotic growth f_n remains an open problem. We conjecture that if $n \equiv 0$ or $3 \pmod 4$, then $\lim_{n\to\infty}(f_n)^{1/n^2} = \sqrt{2}$.

References

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