Solutions for the Practice Final Exam

These problems are sample problems for the final exam, so you may expect similar problems in the final. Do not hand in your solutions. Solutions will be handed out, discussed (and posted on the web) on Tuesday. The final exam is from the material of the whole course, but there will be only one or two problems from the first half. The final exam is a closed book exam, but you may use two sheets of paper (so you may use your midterm sheet) with notes on it.

1. Use the Master Theorem to find the asymptotic solution for the following recurrence: \( T(n) = 5T\left( \frac{n}{2} \right) + n^3 \).

Solution: We have \( a = 5 \), \( b = 2 \),

\[
\frac{f(n)}{n \log_2 5} = \frac{n^3}{n \log_2 5},
\]

we get Case 3, and thus \( T(n) = \Theta(n^3) \). Note that the regularity condition is satisfied as \( 5(\frac{n}{2})^3 = 5n^3/8 \leq cn^3 \) for \( c = 5/8 \). (10 points)

2. Use indicator random variables to find the expected number of bins that remain empty when \( m \) balls are distributed into \( n \) bins uniformly at random. (15 points)

Solution: Let

\[
X_i = \begin{cases} 
1 & \text{if the } i\text{th bin remains empty} \\
0 & \text{otherwise}
\end{cases}
\]

Then by the linearity of expectation,

\[
E(X) = E\left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \left( \frac{n-1}{n} \right)^m = \frac{(n-1)^m}{n^{m-1}}.
\]

3. Show how QUICKSORT can be made to run in \( O(n \log n) \) time in the worst case, assuming that all elements are distinct.
**Solution:** We may use the worst-case linear time Selection algorithm to find the (lower) median, and then we modify QUICKSORT by partitioning around this element. Then QUICKSORT will always recurse on subarrays that are at most half the size of the original array, and thus the recurrence for the worst-case running time is

\[ T(n) \leq 2T(n/2) + \Theta(n) = O(n \log n). \]

(15 points)

4. **Prove or give a counterexample:** For any graph \( G \) with distinct positive weights associated with each edge and for any cut of \( G \), the minimum spanning tree of \( G \) contains exactly one edge belonging to the cut.

**Solution:** This is false. Here is a simple counterexample:

![Graph](image)

Consider the cut \( \{\{a, c\}, \{b, d\}\} \). The MST contains both edges of the cut. (15 points)

5. **Prove or give a counterexample:** Let \( G \) be a flow network with directed edges and positive capacities associated with each edge and let \( (S, T) \) be a minimum cut of the network. If we increase the capacity of every edge in \( G \) by 1 the cut \( (S, T) \) is still a minimum cut.

**Solution:** This is false. Here is a simple counterexample:

![Flow Network](image)
In this flow network consider two minimum cuts, \((\{s, a, b\}, \{c, t\})\) and \((\{s, a, b, c\}, \{t\})\). Increasing the capacity of every edge by 1 increases the capacity of the first cut by 2 and the capacity of the second cut by 1, so the first cut is not minimum anymore. (15 points)

6. What is the least significant decimal digit of \(2^{400}\)?

**Solution:** The remainders \(mod\ 10\) of the powers of 2 follow the pattern 2, 4, 8, 6, you can prove this by induction. Since 100 is divisible by 4, the answer is 6. (15 points)

7. The longest-cycle problem is the problem of determining a cycle of maximum length in a graph \(G\). Formulate a related decision problem, and show that the decision problem is \(NP\)-complete. You may use any of the \(NP\)-complete problems learned in class for your reduction.

**Solution:** Here is the corresponding decision problem:

\[
\text{CYCLE} = \{(G, k) : G \text{ has a cycle of length } k\}.
\]

First we show that CYCLE is in \(NP\). Suppose we are given a graph \(G = (V, E)\) and an integer \(k\). The certificate we choose is the cycle itself. The verification algorithm affirms that this is a cycle of length \(k\). To show that the problem is \(NP\)-hard, we will show

\[
\text{HAM-CYCLE} \leq_P \text{CYCLE}.
\]

The reduction takes as input an instance \(G = (V, E)\) of the Hamiltonian problem and outputs an instance \((G, |V|)\) of the cycle problem. This is indeed a reduction: the graph \(G\) has a Hamiltonian cycle if and only if it has a cycle of length \(|V|\). (15 points)