Homework 1, due Wednesday, September 13

Most homeworks will be worth 100 points; consider the point value in determining how much time you spend on each question. The exercise and page numbers are from the 3rd edition of Cormen, Leiserson, Rivest, Stein: Introduction to Algorithms. Homeworks must be legible and stapled, with writing on only one side of each piece of paper. Homeworks will be collected at the end of lecture on the due date. Grading complaints policy is in the syllabus. This homework assignment and all other handouts are also posted on my home page (URL: http://web.cs.wpi.edu/~gsarkozy).

READING: Chapters 1-5 and Appendix.

1. Find the least integer $k$ such that $f(n)$ is $O(n^k)$ for each of the following functions:
   
   (a) $f(n) = 2n^2 + n^3 \log n$
   
   (b) $f(n) = 3n^5 + (\log n)^4$
   
   (c) $f(n) = (n^4 + n^2 + 1)/(n^4 + 1)$
   
   (d) $f(n) = (n^3 + 5 \log n)/(n^4 + 1)$ (20 points)

2. You have 81 quarters and a balance. You know that 80 quarters have the same weight, and one weighs less than the others. Give an algorithm (in pseudocode) to identify the light quarter which uses the balance only 4 times in the worst case. (20 points)

3. Suppose that we are given a sorted array of distinct integers $A[1, \ldots, n]$ and we want to decide whether there is an index $i$ for which $A[i] = i$.
   
   (a) Describe a divide-and-conquer algorithm that solves this problem.
(b) Use the Master Theorem to estimate the running time of the algorithm you described in part (a). Your algorithm should run in \(O(\log n)\) time. (20 points)

4. Assume that our sample space is the set of permutations of the first \(n\) positive integers \((1, 2, \ldots, n)\), and assume that each permutation is equally likely (we have a uniform random permutation). What is the probability that a random permutation has (a) \(n(n-1)/2\) inversions? (b) 0 inversions? (c) exactly 1 inversion? (d) What is the expected number of inversions in a random permutation? (An inversion is a pair \((i, j)\) where \(i < j\) yet \(j\) is ahead of \(i\)). (20 points)

5. Use indicator random variables to find the expected number of balls that fall into the first bin when \(m\) balls are distributed into \(n\) bins uniformly at random. (20 points)