Solutions for Homework 1

1. If a problem $P$ has worst-case time complexity $\Omega(n \log n)$ and worst-case time complexity $O(n^2)$ and algorithm $A$ solves problem $P$, which of the following is possible?

   (a) $A$ has best-case time complexity $\Theta(n)$.
   (b) $A$ has worst-case time complexity $\Theta(n \sqrt{n})$.
   (c) $A$ has average-case time complexity $\Theta(n^3)$.
   (d) $A$ has worst-case time complexity $\Theta(n)$.

   Solution: (a) possible, (b) possible, (c) possible, (d) impossible (violation of worst-case time complexity $\Omega(n \log n)$). (20 points)

2. You have 81 quarters and a balance. You know that 80 quarters have the same weight, and one weighs less than the others. Give an algorithm (in pseudocode) to identify the light quarter which uses the balance only 4 times in the worst case.

   Solution:

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   for $k = 1$ to 4
     put $81/3^k$ quarters on one pan and $81/3^k$ quarters on the other pan
     if the two pans weigh the same
       then throw away the coins on the balance
     else if the left pan weighs less than the right pan
       then throw away the coins not on the balance and on the right pan
     else throw away the coins not on the balance and on the left pan
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   (20 points)

3. Suppose that each person in a group of $n$ people votes for exactly two people from a set of candidates to fill two positions on a committee.
The top two finishers both win positions as long as each receives more than \( n/2 \) votes.

(a) Describe a divide-and-conquer algorithm that determines the top two candidates and whether these two candidates received more than \( n/2 \) votes.

(b) Use the Master Theorem to estimate the running time of the algorithm you described in part (a). (20 points)

Solution:

(a) Our algorithm will take a sequence of \( 2n \) names (two different names provided by each of \( n \) voters) and determine whether the two top vote-getters occur on our list more than \( n/2 \) times, and if so, who they are. Actually for technical reasons we will need the top 3 vote-getters. The votes of each voter are adjacent. Note that we can have at most 3 people (but not 4) with more than half of the votes. Divide the list into two parts, the first half and the second half. (No one could have gotten more than \( n/2 \) votes on this list without having more than half votes in one half or the other, since if a candidate got less than or equal to half the votes in each half, then he got less than or equal to half the votes in all.) Thus apply the algorithm recursively to each half to come up with at most six names (three from each half). Then run through the entire list to count the number of occurrences of each of these names to decide which, if any, are the winners. This requires at most \( 12n \) additional comparisons for a list of length \( 2n \), thus for the total number of comparisons we get the following recursion:

\[
T(n) = 2T\left(\frac{n}{2}\right) + 12n.
\]

(b) By the Master Theorem, since we have \( a = 2 \), \( b = 2 \),

\[
f(n) = 12n = \Theta(n^{\log_2{2}}) = \Theta(n),
\]

we get Case 2, and thus \( T(n) = \Theta(n \log n) \). (20 points)
4. Assume that our sample space is the set of permutations of the first $n$ positive integers $(1, 2, \ldots, n)$, and assume that each permutation is equally likely (we have a uniform random permutation). What is the probability that a random permutation has (a) $n(n - 1)/2$ inversions? (b) 0 inversions? (c) exactly 1 inversion?

**Solution:**

(a) The random permutation has $n(n - 1)/2$ inversions exactly when it is sorted in decreasing order. Since exactly one of the $n!$ permutations is sorted in decreasing order, its probability is $1/n!$.

(b) The random permutation has 0 inversions exactly when it is sorted in increasing order, its probability is again $1/n!$.

(c) The random permutation has exactly 1 inversion in the following $(n - 1)$ permutations: $(2, 1, 3, 4, \ldots, n), \ (1, 3, 2, 4, \ldots, n), \ldots, \ (1, 2, 3, \ldots, n, n - 1)$. The probability of drawing one of these permutations is $(n - 1)/n!$. (20 points)

5. We are given the same sample space as in the previous problem. What is the expected number of fixed elements, that is, elements left in the same position, in a random permutation?

**Solution:** Let

$$X_i = \begin{cases} 
1 & \text{if the } i\text{th element is fixed} \\
0 & \text{otherwise}
\end{cases}$$

Then by the linearity of expectation

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{1}{n} = \frac{n}{n} = 1.$$ 

(20 points)