

CS 3133 Foundations of Computer Science
C term 2012

Solutions for Homework 2

READING: Chapters 3, 4, 5, 18.

1. Exercise 2 on page 97.

Solution:

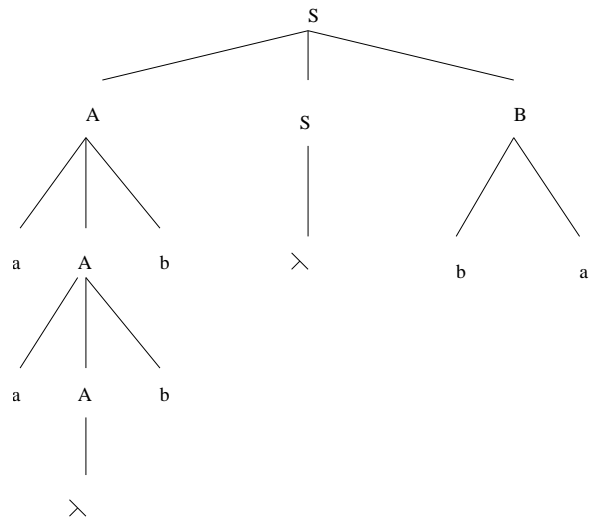
- (a) The following is a leftmost derivation of $aabbba$:

$$\begin{aligned} S &\Rightarrow ASB \\ &\Rightarrow aAbSB \\ &\Rightarrow aaAbbSB \\ &\Rightarrow aabbSB \\ &\Rightarrow aabbB \\ &\Rightarrow aabbba \end{aligned}$$

- (b) The following is a rightmost derivation of $abaabbbabbaa$:

$$\begin{aligned} S &\Rightarrow ASB \\ &\Rightarrow ASbBa \\ &\Rightarrow ASbbaa \\ &\Rightarrow AASBbbaa \\ &\Rightarrow AASbabbbaa \\ &\Rightarrow AAbabbbaa \\ &\Rightarrow AaAbbabbaa \\ &\Rightarrow AaaAbbbabbbaa \\ &\Rightarrow Aaabbbabbbaa \\ &\Rightarrow aAbaabbbabbbaa \\ &\Rightarrow abaabbbabbbaa \end{aligned}$$

(c) Derivation tree for (a):



DT:

$$\begin{aligned} S &\Rightarrow AB \\ &\Rightarrow aAB \\ &\Rightarrow aaB \\ &\Rightarrow aaAB \\ &\Rightarrow aaaB \\ &\Rightarrow aaab \end{aligned}$$

(b) The following is a rightmost derivation that generates the given tree DT:

$$\begin{aligned} S &\Rightarrow AB \\ &\Rightarrow AAB \\ &\Rightarrow AAb \\ &\Rightarrow Aab \\ &\Rightarrow aAab \\ &\Rightarrow aaab \end{aligned}$$

(c) There are 20 derivations that generate DT. (20 points)

3. Exercise 7 on page 98.

Solution: The following is a grammar over $\{a, b, c\}$ whose language is exactly $\{a^n b^{2n} c^m \mid n, m > 0\}$:

$$\begin{aligned} S &\rightarrow AC \\ A &\rightarrow aAbb \mid abb \\ C &\rightarrow Cc \mid c \end{aligned}$$

(15 points)

4. Show by induction that for every natural number n , 3 is a divisor of $n^3 + 2n$.

Solution: Basis: It is true for $n = 0$.

Inductive Hypothesis: Assume that it is true for all values $k = 0, 1, \dots, n$, i.e.

$$3 \mid k^3 + 2k.$$

Inductive Step: We need to show that it is true for $n + 1$, i.e.

$$3 \mid (n + 1)^3 + 2(n + 1).$$

Indeed,

$$\begin{aligned}(n+1)^3 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 2n + 2 = \\ &= (n^3 + 2n) + 3n^2 + 3n + 3 = (n^3 + 2n) + 3(n^2 + n + 1),\end{aligned}$$

which is divisible by three. (The first part because of the Inductive Hypothesis and the second part trivially.) (15 points)

5. Let G be the grammar

$$\begin{aligned}S &\rightarrow ASB|\lambda \\ A &\rightarrow a \\ B &\rightarrow b.\end{aligned}$$

- (a) What is $L(G)$?
- (b) Prove formally (so using induction on the length of the derivations) that $L(G)$ is the set given in (a).

Solution:

(a) $L(G) = \{a^n b^n | n \geq 0\}$.

(b) First we show that $L(G) \subseteq \{a^n b^n | n \geq 0\}$. For this purpose we will show by induction on the length of the derivations that $n(a) + n(A) = n(b) + n(B)$ and that in all the strings in the derivation the a -s and A -s form a prefix of the string and the b -s and B -s form a suffix of the string.

Basis: Derivations of length 0, so S . True. Remark: Sometimes you have to start with derivations of length 1 as basis. Here the statement is true for $n = 0$ so we can start with $n = 0$ as basis, but sometimes this is not case.

Inductive Hypothesis: We assume that this statement is true for all strings w that can be obtained by n rule applications, so $S \xrightarrow{n} w$.

Inductive Step: We have to show that the statement is true for all strings w that can be obtained by $n + 1$ rule applications, so $S \xrightarrow{n+1} w$. Once again the key step is to reformulate the derivation to apply the inductive hypothesis. The derivation of w can be written $S \xrightarrow{n} w' \Rightarrow w$. By the Inductive Hypothesis we know that the statement is true for w' , so $n_{w'}(a) + n_{w'}(A) = n_{w'}(b) + n_{w'}(B)$ (say = k) and the a -s and A -s

form a prefix of w' and the b -s and B -s form a suffix of w' . But these obviously remain true when we apply one more rule:

Rule	$n_w(a) + n_w(A)$	$n_w(b) + n_w(B)$
$S \rightarrow ASB$	$k + 1$	$k + 1$
$S \rightarrow \lambda$	k	k
$A \rightarrow a$	k	k
$B \rightarrow b$	k	k

Next we show that $\{a^n b^n | n \geq 0\} \subseteq L(G)$. Indeed,

$$\begin{aligned}
 S &\xrightarrow{n} \underbrace{A \dots A}_n S \underbrace{B \dots B}_n \\
 &\Rightarrow \underbrace{A \dots A}_n \underbrace{B \dots B}_n \\
 &\xrightarrow{n} \underbrace{a \dots a}_n \underbrace{B \dots B}_n \\
 &\xrightarrow{n} \underbrace{a \dots a}_n \underbrace{b \dots b}_n
 \end{aligned}$$

(20 points)

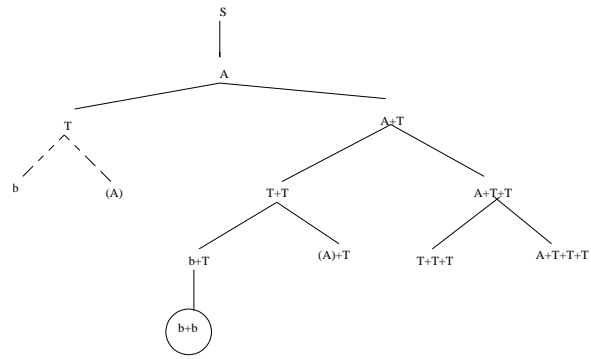
6. In this problem we consider the grammar of arithmetic expressions AE , so

$$\begin{aligned}
 AE : \quad V &= \{S, A, T\} \\
 \Sigma &= \{b, +, (,)\} \\
 P : \quad 1. &S \rightarrow A \\
 &2. A \rightarrow T \\
 &3. A \rightarrow A + T \\
 &4. T \rightarrow b \\
 &5. T \rightarrow (A)
 \end{aligned}$$

Build the search tree constructed by the breadth-first top-down parsing algorithm for the string $b + b$.

Solution:

The breadth-first top-down search tree:



(15 points)