CS 3133 Foundations of Computer Science  
A term 2019

Solutions for Homework 2

READING: Chapters 3, 4, 5, 18.

1. Exercise 2 on page 97.

Solution:

(a) The following is a leftmost derivation of $aabbba$:

$$
S \Rightarrow ASB \\
  \Rightarrow aAbSB \\
  \Rightarrow aaAbbSB \\
  \Rightarrow aabbSB \\
  \Rightarrow aabbB \\
  \Rightarrow aabbba
$$

(b) The following is a rightmost derivation of $abaabbbabbaa$:

$$
S \Rightarrow ASB \\
  \Rightarrow ASbBa \\
  \Rightarrow ASbbaa \\
  \Rightarrow AASBbaa \\
  \Rightarrow AASbabbaa \\
  \Rightarrow AAbabbaa \\
  \Rightarrow AaAbbbaaa \\
  \Rightarrow AaaAbbbabbaa \\
  \Rightarrow Aaabbbabbaa \\
  \Rightarrow aAbaabbabbaa \\
  \Rightarrow abaabbbabbaa
$$
(c) Derivation tree for (a):
Derivation tree for (b):

(d) \[ L(G) = \{a^{n_i}b^{n_k}b^{m_1}a^{m_1} \ldots b^{m_l}a^{m_l}|n_i, m_j, l \geq 0, k \geq 0, k \leq l\} \]

(15 points)

2. Exercise 4 on page 98.

Solution:

(a) The following is a leftmost derivation that generates the given tree
DT:

\[
\begin{align*}
S & \Rightarrow AB \\
& \Rightarrow aAB \\
& \Rightarrow aaB \\
& \Rightarrow aaAB \\
& \Rightarrow aaaB \\
& \Rightarrow aaab
\end{align*}
\]

(b) The following is a rightmost derivation that generates the given tree DT:

\[
\begin{align*}
S & \Rightarrow AB \\
& \Rightarrow AAB \\
& \Rightarrow AAb \\
& \Rightarrow Aab \\
& \Rightarrow aAab \\
& \Rightarrow aaab
\end{align*}
\]

(c) There are 20 derivations that generate DT. (20 points)


Solution: The following is a grammar over \( \{a, b\} \) whose language is exactly \( \{a^m b^i a^n \mid i = m + n\} \):

\[
\begin{align*}
S & \rightarrow AB \mid \lambda \\
A & \rightarrow aAb \mid \lambda \\
B & \rightarrow bBa \mid \lambda
\end{align*}
\]

(15 points)

4. Show by induction that for every natural number \( n \)

\[
0^2 + 1^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}.
\]

Solution:

Basis: It is true for \( n = 0 \).

Inductive Hypothesis: Assume that it is true for all values \( k = 0, 1, \ldots, n \), i.e.

\[
\sum_{i=0}^{k} i^2 = \frac{k(k + 1)(2k + 1)}{6}.
\]
Inductive Step: We need to show that it is true for \( n + 1 \), i.e.

\[
\sum_{i=0}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}.
\]

Indeed,

\[
\sum_{i=0}^{n+1} i^2 = \sum_{i=0}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \\
= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)(2n^2 + 7n + 6)}{6} = \\
= \frac{(n+1)(n+2)(2n+3)}{6}.
\]

(15 points)

5. Let \( G \) be the grammar
\[
S \to ASB|\lambda \\
A \to a \\
B \to b.
\]

(a) What is \( L(G) \)?

(b) Prove formally (so using induction on the length of the derivations) that \( L(G) \) is the set given in (a).

Solution:

(a) \( L(G) = \{ a^n b^n | n \geq 0 \} \).

(b) First we show that \( L(G) \subset \{ a^n b^n | n \geq 0 \} \). For this purpose we will show by induction on the length of the derivations that \( n(a) + n(A) = n(b) + n(B) \) and that in all the strings in the derivation the \( a \)-s and \( A \)-s form a prefix of the string and the \( b \)-s and \( B \)-s form a suffix of the string.

Basis: Derivations of length 0, so \( S \). True. Remark: Sometimes you have to start with derivations of length 1 as basis. Here the statement is true for \( n = 0 \) so we can start with \( n = 0 \) as basis, but sometimes this is not case.
Inductive Hypothesis: We assume that this statement is true for all strings \( w \) that can be obtained by \( n \) rule applications, so \( S \xrightarrow{n} w \).

Inductive Step: We have to show that the statement is true for all strings \( w \) that can be obtained by \( n+1 \) rule applications, so \( S \xrightarrow{n+1} w \). Once again the key step is to reformulate the derivation to apply the inductive hypothesis. The derivation of \( w \) can be written \( S \xrightarrow{n} w' \Rightarrow \Rightarrow w \).

By the Inductive Hypothesis we know that the statement is true for \( w' \), so \( n_{w'}(a) + n_{w'}(A) = n_{w'}(b) + n_{w'}(B) \) (say = \( k \)) and the \( a \)-s and \( A \)-s form a prefix of \( w' \) and the \( b \)-s and \( B \)-s form a suffix of \( w' \). But these obviously remain true when we apply one more rule:

\[
\begin{align*}
\text{Rule} & \quad n_{w}(a) + n_{w}(A) & n_{w}(b) + n_{w}(B) \\
S \rightarrow ASB & \quad k+1 & k+1 \\
S \rightarrow \lambda & \quad k & k \\
A \rightarrow a & \quad k & k \\
B \rightarrow b & \quad k & k
\end{align*}
\]

Next we show that \( \{a^n b^n | n \geq 0\} \subset L(G) \). Indeed,

\[
\begin{align*}
S & \xrightarrow{n} A \ldots ASB \ldots B \\
& \Rightarrow A \ldots A B \ldots B \\
& \xrightarrow{n} a \ldots a B \ldots B \\
& \xrightarrow{n} a \ldots a b \ldots b
\end{align*}
\]

(20 points)

6. In this problem we consider the grammar of arithmetic expressions \( AE \), so

\[
\begin{align*}
AE : & \quad V = \{S, A, T\} \\
& \quad \Sigma = \{b, +, (, )\} \\
P : & \quad 1. S \rightarrow A \\
& \quad 2. A \rightarrow T \\
& \quad 3. A \rightarrow A + T \\
& \quad 4. T \rightarrow b \\
& \quad 5. T \rightarrow (A)
\end{align*}
\]
Build the search tree constructed by the breadth-first top-down parsing algorithm for the string $b + b$.

**Solution:**

The breadth-first top-down search tree:

![Breadth-first top-down search tree diagram]

(15 points)