

CS 2022/ MA 2201 Discrete Mathematics
A term 2011

Solutions for the Midterm Exam

1. Are $\neg(p \oplus q)$ and $p \leftrightarrow q$ logically equivalent? Justify your response.

Solution: Yes, here is the truth table.

p	q	$\neg(p \oplus q)$	$p \leftrightarrow q$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

(20 points)

2. Let $I(x)$ be the statement “ x has an Internet connection”, where the universe consists of all students in the class. Express the following in predicate logic: “Everyone except one student in the class has an Internet connection.”

Solution: Here is the predicate logic formula

$$\exists x \forall y (x \neq y \leftrightarrow I(y)).$$

(20 points)

3. Give an example of a function $f : \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$ (where \mathbf{N} denotes the set of natural numbers and \times is the Cartesian product) that is
- neither one-to-one nor onto,
 - onto but not one-to-one,
 - one-to-one but not onto.

Solution: a.) $f(m, n) = 2022$ (any constant function). b.) $f(m, n) = m+n$, we can get every natural number n as an image, since for example $f(0, n) = n$. c.) $f(m, n) = 2^m 3^n$, this is not onto since we never get 5 for example, and it is one-to-one as $2^{m_1} 3^{n_1} = 2^{m_2} 3^{n_2}$ implies $m_1 = m_2$ and $n_1 = n_2$ by the Fundamental Theorem of Arithmetic. (20 points)

4. (a) Translate the following inference into propositional logic.
 If the portrait resembles the customer, the artist will be disappointed. If the portrait does not resemble the customer, his wife will refuse to pay. If the customer's wife refuses to pay, the artist will be disappointed. Therefore the artist will be disappointed.
- (b) Is the inference correct? Justify your response.

Solution: (a) p - The portrait resembles the customer, d - The artist will be disappointed, r - The customer's wife refuses to pay. The argument is:

$$\begin{array}{l} p \rightarrow d \\ \neg p \rightarrow r \\ r \rightarrow d \\ \hline d \end{array}$$

(b) Yes. From $\neg p \rightarrow r$ and $r \rightarrow d$ we can conclude $\neg p \rightarrow d$ (hypothetical syllogism). From $p \rightarrow d$, $\neg p \rightarrow d$ and $p \vee \neg p$ (this is always true) we can conclude d (dilemma or proof by cases). (20 points)

5. Let $x \geq 0$ be an arbitrary non-negative real number. Are the following two propositions equivalent? (a) x is irrational, (b) $\sqrt{3x}$ is irrational. If yes, explain, if not, then what is the relationship between the two statements?

Solution: They are *not* equivalent. (a) \rightarrow (b) is true, but (b) \rightarrow (a) is not true in general. To see that (b) \rightarrow (a) is not true in general, we can have the counterexample $x = 2/3$. We can use a proof by contraposition to prove (a) \rightarrow (b). We have to prove the contrapositive of the statement: If $\sqrt{3x}$ is rational, then x is rational. Assume that $\sqrt{3x}$ is rational, so $\sqrt{3x} = \frac{p}{q}$ for some integers p and q . We have to show that in this case x is rational as well. Now taking the square

$$3x = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2},$$

so

$$x = \frac{p^2}{3q^2}$$

a fraction of two integers, so x is rational, as we wanted. (20 points)