CS 2022/ MA 2201 Discrete Mathematics
A term 2015

Solutions for the Midterm Exam

1. Are $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ logically equivalent? Justify your response.

   Solution: No, for example if they are all false, then the first one is false but the second one is true. You can also get this from the truth table. (20 points)

2. Express the following in predicate logic: "Every student in this class has taken some course in every department of the university."

   Solution: Let $T(s, c, d)$ be the statement that student s has taken class c in department d. The universes are the students in this class, the courses at the university and the departments at the university. Here is the predicate logic formula

   $$\forall s \forall d \exists c \; T(s, c, d).$$

   (20 points)

3. Give an example of a function f from the set of integers to the set of positive even integers that is

   (a) one-to-one, but not onto,

   (b) onto, but not one-to-one,

   (c) one-to-one and onto,

   (d) neither one-to-one nor onto.

   Solution: a.) There are many good solutions. One good solution is the following: $f(n) = 4n + 4$ if $n > 0$ and $f(n) = 4|n| + 2$ if $n \leq 0$ (we never get 4). b.) $f(n) = 2|n| + 2$ for any integer n. c.) $f(n) = 4n$ if $n > 0$ and $f(n) = 4|n| + 2$ if $n \leq 0$. d.) $f(n) = 2022$ (any constant function). (20 points)

4. (a) Translate the following inference into propositional logic.
   If the portrait resembles the customer, the artist will be disap-
   pointed. If the portrait does not resemble the customer, his wife
   will refuse to pay. If the customer's wife refuses to pay, the artist
   will be disappointed. Therefore the artist will be disappointed.

   (b) Is the inference correct? Justify your response.

   Solution: (a) p - The portrait resembles the customer, d - The artist
   will be disappointed, r - The customer's wife refuses to pay. The argu-
   ment is:

   $$p \rightarrow d$$
   $$\neg p \rightarrow r$$
   $$\frac{r \rightarrow d}{d}$$

   (b) Yes. From $\neg p \rightarrow r$ and $r \rightarrow d$ we can conclude $\neg p \rightarrow d$ (hypothet-
   ical syllogism). From $p \rightarrow d$, $\neg p \rightarrow d$ and $p \vee \neg p$ (this is always true)
   we can conclude d (dilemma or proof by cases). (20 points)

5. Let $x \geq 0$ be an arbitrary non-negative real number. Are the following
   two propositions equivalent? (a) x is irrational, (b) x^2 is irrational. If
   yes, explain, if not, then what is the logical relationship between the
   two statements?

   Solution: They are *not* equivalent. (b)→(a) is true, but (a)→(b) is not
   true in general. To see that (a)→(b) is not true in general, we can have
   the counterexample $x = \sqrt{2}$. We can use a proof by contraposition to
   prove (b)→(a). We have to prove the contrapositive of the statement:
   If x is rational, then x^2 is rational. Assume that x is rational, so $x = \frac{p}{q}$
   for some integers p and q. We have to show that in this case x^2 is
   rational as well. Now taking the square

   $$x^2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2},$$

   a fraction of two integers, so x^2 is rational, as we wanted.(20 points)