CS 2022/ MA 2201 Discrete Mathematics
A term 2009

Solutions for Homework 5

1. Exercise 16 on page 399.

   Solution: There are $C(52,5) = \binom{52}{5} = 2,598,960$ different hands. There are $4 \cdot C(13,5) = 4 \cdot \binom{13}{5}$ ways to choose the flush by choosing the suit first and then choosing 5 cards from that suit. Thus the probability is

   $$\frac{4 \cdot \binom{13}{5}}{\binom{52}{5}} = \frac{5,148}{2,598,960}.$$

   (15 points)

2. In an experiment you pick at random a bit string of length 5. Consider the following events: E_1: the bit string chosen begins with 1, E_2: the bit string chosen ends with 1, E_3: the bit string chosen has exactly three 1s.

   (a) Find $p(E_1|E_3)$.
   (b) Find $p(E_3|E_2)$.
   (c) Find $p(E_2|E_3)$.
   (d) Find $p(E_3|E_1 \cap E_2)$.
   (e) Determine whether E_1 and E_2 are independent.
   (f) Determine whether E_2 and E_3 are independent.

   Solution: We will need the following probabilities:

   $$p(E_1) = p(E_2) = \frac{1}{2}, p(E_3) = \frac{\binom{5}{3}}{2^5} = \frac{5}{16},$$

   $$p(E_1 \cap E_2) = \frac{1}{4}, p(E_1 \cap E_3) = p(E_2 \cap E_3) = \frac{\binom{4}{2}}{2^5} = \frac{3}{16},$$

$$p(E_1 \cap E_2 \cap E_3) = \frac{3}{2^5} = \frac{3}{32}.$$

Then we have

a.) $p(E_1|E_3) = \frac{p(E_1 \cap E_3)}{p(E_3)} = \frac{\frac{3}{16}}{\frac{5}{16}} = \frac{3}{5}$.

b.) $p(E_3|E_2) = \frac{p(E_2 \cap E_3)}{p(E_2)} = \frac{\frac{3}{16}}{\frac{1}{2}} = \frac{3}{8}$.

c.) $p(E_2|E_3) = \frac{p(E_2 \cap E_3)}{p(E_3)} = \frac{\frac{3}{16}}{\frac{5}{16}} = \frac{3}{5}$.

d.) $p(E_3|E_1 \cap E_2) = \frac{p(E_1 \cap E_2 \cap E_3)}{p(E_1 \cap E_2)} = \frac{\frac{3}{32}}{\frac{1}{4}} = \frac{3}{8}$.

e.) They are independent, since

$$p(E_1 \cap E_2) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = p(E_1) \cdot p(E_2).$$

f.) They are not independent, since

$$p(E_2 \cap E_3) = \frac{3}{16} \neq \frac{1}{2} \cdot \frac{5}{16} = p(E_2) \cdot p(E_3).$$

(20 points)

3. Exercise 34 on page 416.

Solution: We need to use the binomial distribution, which tells us that the probability of k successes is

$$b(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}.$$

a.) Here $k = 0$, since we want all the trials to result in failure. Thus we get $b(0; n, p) = 1 \cdot p^0 \cdot (1-p)^n = (1-p)^n$.

b.) There is at least one success if and only if it is not the case that there are no successes. Thus we obtain the answer by subtracting the probability in part a.) from 1, namely $1 - (1-p)^n$.

c.) There are two ways in which there can be at most one success: no successes or one success. The probability of no success is $(1-p)^n$.

The probability of one success is $np(1-p)^{n-1}$. Therefore the answer is $(1-p)^n + np(1-p)^{n-1}$.

d.) Since this event is just that the event in part c.) does not happen, the answer is $1 - ((1-p)^n + np(1-p)^{n-1})$. (20 points)

4. Exercise 16 on page 440.

   Solution: We need to show that $p(X = i \text{ and } Y = j)$ is not always equal to $p(X = i)p(Y = j)$. Here is a counterexample: $i = j = 2$. $p(X = 2 \text{ and } Y = 2)$ is equal to 0, while $p(X = 2) = p(Y = 2) = \frac{1}{4}$, so $p(X = 2)p(Y = 2) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$. (15 points)

5. What is the expected value, variance and standard deviation of the number of times a 6 appears when a fair die is rolled 30 times?

   Solution: The expected value, variance and standard deviation of the number of successes in n Bernoulli trials are np, npq, $\sqrt{npq}$, resp. Here $n = 30$, $p = \frac{1}{6}$ and $q = \frac{5}{6}$. Therefore the expected value is $\frac{30}{6} = 5$, the variance is $\frac{150}{36} = \frac{25}{6}$, and the standard deviation is $\sqrt{\frac{25}{6}} = \frac{5}{\sqrt{6}}$. (15 points)

6. Exercise 6.a-d) on page 528.

   Solution: a.) Since $1 + 1 \neq 0$, this relation is not reflexive. Since $x + y = y + x$, it follows that $x + y = 0$ if and only if $y + x = 0$, so the relation is symmetric. Since $(1, -1)$ and $(-1, 1)$ are both in R, the relation is not antisymmetric. The relation is not transitive; for example, $(1, -1) \in R$ and $(-1, 1) \in R$, but $(1, 1) \notin R$.

   b.) The relation is reflexive, since $x = x$ (choosing the + sign). The relation is symmetric, because $x = \pm y$ if and only if $y = \pm x$. Since $(1, -1)$ and $(-1, 1)$ are both in R, the relation is not antisymmetric. To see that the relation is transitive, note that the product of 1's and -1's is ± 1.

   c.) The relation is reflexive, since $x - x = 0$ is a rational number. The relation is symmetric, because if $x - y$ is rational, then so is $-(x - y) = y - x$. Since $(1, -1)$ and $(-1, 1)$ are both in R, the relation is not antisymmetric. To see that the relation is transitive, note that if $(x, y) \in R$ and $(y, z) \in R$, then $x - y$ and $y - z$ are rational numbers. Therefore their sum $x - z$ is rational, and that means that $(x, z) \in R$.

d.) The relation is not reflexive since $1 \neq 2 \cdot 1$. It is not symmetric, since $(2,1) \in R$, but $(1,2) \notin R$. To see that it is antisymmetric, suppose that $x = 2y$ and $y = 2x$. Then $x = 4x$ which implies $x = 0$ and hence $y = 0$. Thus the only time that (x, y) and (y, x) are both in R is when $x = y$ (and both are 0). It is clearly not transitive; for example, $(4,2) \in R$ and $(2,1) \in R$, but $(4,1) \notin R$. (15 points)