CS 2022/ MA 2201 Discrete Mathematics
A term 2015

Solutions for Homework 4

1. Exercise 14 on page 330.

   Solution:

   Basis Step: For $n = 1$ we get $1 \cdot 2^1 = (1-1)2^{1+1} + 2$, which is true statement $2 = 2$.

   Inductive Step: Let us assume the inductive hypothesis, namely that

   $$\sum_{k=1}^{n} k \cdot 2^k = (n-1)2^{n+1} + 2,$$

   and then from this we have to prove that

   $$\sum_{k=1}^{n+1} k \cdot 2^k = n \cdot 2^{n+2} + 2.$$

   Indeed, by using the inductive hypothesis we get

   $$\sum_{k=1}^{n+1} k \cdot 2^k = \left(\sum_{k=1}^{n} k \cdot 2^k\right) + (n+1)2^{n+1} = (n-1)2^{n+1} + 2 + (n+1)2^{n+1} =$$

   $$= 2n \cdot 2^{n+1} + 2 = n \cdot 2^{n+2} + 2,$$

   as desired. (15 points)

2. Exercise 46 on page 397.

   Solution: (a) We first place the bride in any of the 6 positions. Then, from left to right in the remaining positions, we choose the other 5 people to be in the picture; this can be done in $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15,120$ ways. Therefore, the answer is $6 \cdot 15,120 = 90,720$.

   (b) We first place the bride in any of the 6 positions, and then place the groom in any of the remaining 5 positions. Then, from left to right in the remaining positions, we choose the other 4 people to be in the picture; this can be done in $8 \cdot 7 \cdot 6 \cdot 5 = 1,680$ ways. Therefore, the answer is $6 \cdot 5 \cdot 1,680 = 50,400$.

(c) From part (a) there are $90,720$ ways for the bride to be in the picture. There are (from part (b)) $50,400$ ways for both the bride and groom to be in the picture. Therefore there are $90,720 - 50,400 = 40,320$ ways for just the bride to be in the picture. Symmetrically, there are $40,320$ ways for just the groom to be in the picture. Therefore, the answer is $40,320 + 40,320 = 80,640$.

(15 points)

3. Exercise 26 on page 406.

   Solution: Let the people be A, B, C, D and E. Suppose the following pairs are friends: $(A, B), (B, C), (C, D), (D, E)$ and (E, A). The other 5 pairs are enemies. In this example, there are no three mutual friends and no three mutual enemies. (15 points)

4. Exercise 26 on page 414.

   Solution:

   (a) $C(13, 10) = \binom{13}{10} = 286$.

   (b) $P(13, 10) = 13 \cdot 12 \cdot \ldots 4$.

   (c)

   - First solution: There is only one way to choose the 10 players without choosing a woman, since there are exactly 10 men. Therefore using part a.) there are 286-1=285 ways to choose the players if at least one of them must be a woman.
   - Second Solution: $C(3, 1) \cdot C(10, 9) + C(3, 2) \cdot C(10, 8) + C(3, 3) \cdot C(10, 7) = \binom{3}{1} \cdot \binom{10}{9} + \binom{3}{2} \cdot \binom{10}{8} + \binom{3}{3} \cdot \binom{10}{7} = 285$.

   (20 points)

5. Exercise 8 on page 421.

Solution: By the binomial theorem the coefficient is

$$\binom{17}{9}3^8 2^9.$$

(15 points)

6. Exercise 22 on page 422.

   Solution: Combinatorial argument: On both sides we count the number of ways we can select from n elements two subsets, a subset A with k elements and another, disjoint, subset B with $r-k$ elements.

   Algebraic argument:

$$\binom{n}{r}\binom{r}{k} = \frac{n!}{r!(n-r)!}\cdot\frac{r!}{k!(r-k)!} = \frac{n!}{k!(n-r)!(r-k)!},$$

$$\binom{n}{k}\binom{n-k}{r-k} = \frac{n!}{k!(n-k)!}\cdot\frac{(n-k)!}{(r-k)!(n-r)!} = \frac{n!}{k!(n-r)!(r-k)!}.$$

   (20 points)