CS 2022/ MA 2201 Discrete Mathematics
A term 2015

Solutions for Homework 3

READING: Chapter 1, 2, 3, 4.

1. Exercise 8 on page 216.

   Solution: In each case we have to find the least integer n such that $f(x)$ is $O(x^n)$ so there must be constants C and k such that $|f(x)| \leq C|x^n|$ for $x > k$.

   a) Since $x^3 \log x$ is not $O(x^3)$ (because the $\log x$ factor grows without bound as x increases), $n = 3$ is too small. On the other hand, certainly $\log x$ grows more slowly than x, so $2x^2 + x^3 \log x \leq 2x^4 + x^4 = 3x^4$. Therefore $n = 4$ is the answer, with $C = 3$ and $k = 1$.

   b) The $(\log x)^4$ is insignificant compared to the x^5 term, so the answer is $n = 5$. Formally we can take $C = 4$ and $k = 1$.

   c) For large x, this fraction is close to 1 (this can be seen by dividing the numerator and the denominator by x^4). Therefore the answer is $n = 0$, in other words this function is $O(x^0) = O(1)$. Formally we can write $f(x) \leq 3x^4/x^4 = 3$ for all $x > 1$, so we can take $C = 3$ and $k = 1$.

   d) Here the answer is $n = -1$, since for large x, $f(x)$ is close to $1/x$. Formally we can write $f(x) \leq 6x^3/x^4 = 6/x$ for all $x > 1$, so we can take $C = 6$ and $k = 1$. (15 points)

2. Describe an algorithm for finding the two smallest integers in a finite sequence of distinct integers. What is the worst-case complexity of your algorithm (counting comparisons only)?

   Solution: Say the sequence is $a_1, a_2, \ldots, a_n$. We maintain two integers, the two temporary minimums, min_1 and min_2, where $min_1 < min_2$.

   1. Compare a_1 and a_2. If $a_1 < a_2$ then set min_1 equal to a_1 and set min_2 equal to a_2. If $a_2 < a_1$ then set min_1 equal to a_2 and set min_2 equal to a_1.

2. Compare the next integer $a_i, i \geq 3$ in the sequence to min_1 and min_2. If $a_i < min_1$, then set $min_2 = min_1$ and $min_1 = a_i$. If $min_2 > a_i > min_1$, then set $min_2 = a_i$. Finally, if $a_i > min_2$, then do not change min_1 and min_2.

3. Repeat the previous step if there are more integers left in the sequence.

4. Stop when there are no integers left. min_1 is the smallest integer, and min_2 is the 2nd smallest integer.

The worst-case time complexity is $3(n-2)+2 = 3n-4$ comparisons.

(15 points)

3. Exercise 10 on page 79.

   Solution:

   (a) If we use modus tollens starting from the back, then we conclude that I am not sore. Another application of modus tollens the tells that I did not play hockey.

   (b) We really can't conclude anything specific here.

   (c) By universal instantiation, we conclude from the first conditional statement by modus ponens that dragonflies have six legs, and we conclude by modus tollens that spiders are not insects. We could say using existential generalization that, for example, there exists a non-six-legged creature that eats a six-legged creature, and that there exists a non-insect that eats an insect.

   (d) We can apply universal instantiation to the conditional statement and conclude that if Homer (respectively, Maggie) is a student, then he (she) has an internet account. Now modus tollens tells us that Homer is not a student. There are no conclusions to be drawn about Maggie.

   (e) The first conditional statement is that if x is healthy to eat, then x does not taste good. Universal instantiation and modus ponens therefore tell us that tofu does not taste good. The third sentence says that if you eat x, then x tastes good. Therefore the fourth hypothesis already follows (by modus tollens) from the first three. No conclusions can be drawn about cheeseburgers from these statements.

(f) By disjunctive syllogism, the first two hypoteses allow us to conclude that I am hallucinating. Therefore by modus ponens we know that I see elephants running down the road.

(20 points)

4. Exercise 20 on page 80.

   Solution:

   (a) Invalid, it is the fallacy of affirming the conclusion. Letting $a = -2$ provides a counterexample.

   (b) Valid, it is modus ponens.

   (15 points)

5. Exercise 12 on page 91.

   Solution: This is true. Suppose that a/b is a non-zero rational number and that x is an irrational number. We must prove that the product xa/b is also irrational. We give a proof by contradiction. Suppose that xa/b were rational. Since $a/b \neq 0$, we know that $a \neq 0$, so b/a is also a rational number. Let us multiply this rational number b/a by the assumed rational number xa/b. The product of two rational numbers is rational. But the product is $(b/a)(xa/b) = x$, which is irrational by hypothesis. This is a contradiction, so in fact xa/b must be irrational, as desired. (15 points)

6. Exercise 16 on page 91.

   Solution: We give a proof by contraposition. If it is not true that m is even or n is even, than m and n are both odd, say $m = 2k + 1$ and $n = 2l + 1$ for some integers k and l. But then $mn = (2k+1)(2l+1) = 2(2kl + k + l) + 1$ is odd, and our proof is complete. (20 points)