Computer Graphics (CS 543)
Lecture 6 (Part 3): Derivation of Perspective Projection Transformation

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Perspective Projection

- Projection – map the object from 3D space to 2D screen

Perspective()  Frustum()
Perspective Projection: Classical

Based on similar triangles:

\[ \frac{y'}{y} = \frac{N}{-z} \]

\[ y' = y \cdot \frac{N}{-z} \]
Perspective Projection: Classical

- So \((x^*, y^*)\) projection of point, \((x,y,z)\) unto near plane \(N\) is given as:

\[
(x^*, y^*) = \left( \frac{x}{-z}, \frac{y}{-z} \right)
\]

- Numerical example:

Q. Where on the viewplane does \(P = (1, 0.5, -1.5)\) lie for a near plane at \(N = 1\)?

\[
(x^*, y^*) = \left( \frac{x}{-z}, \frac{y}{-z} \right) = \left( \frac{1}{1.5}, \frac{0.5}{1.5} \right) = (0.666, 0.333)
\]
Pseudodepth

- Classical perspective projection projects \((x, y)\) coordinates to \((x^*, y^*)\), drops \(z\) coordinates

- But we need \(z\) to find closest object (depth testing)!!!
Perspective Transformation

- **Perspective transformation** maps actual z distance of perspective view volume to range \([-1 \text{ to } 1]\) (Pseudodepth) for canonical view volume.

We want perspective Transformation and **NOT** classical projection!!

Set scaling z
Pseudodepth = az + b
Next solve for a and b
Perspective Transformation

- We want to transform viewing frustum volume into canonical view volume.

Canonical View Volume

((1, 1, -1), (-1, -1, 1))

Canonical View Volume
Perspective Transformation using Pseudodepth

\[(x^*, y^*, z^*) = \left(\frac{x}{N - z}, \frac{y}{N - z}, \frac{az + b}{-z}\right)\]

- Choose \(a, b\) so as \(z\) varies from Near to Far, pseudodepth varies from \(-1\) to \(1\) (canonical cube)

- Boundary conditions
  - \(z^* = -1\) when \(z = -N\)
  - \(z^* = 1\) when \(z = -F\)
Transformation of $z$: Solve for $a$ and $b$

- **Solving:**
  \[ z^* = \frac{az + b}{-z} \]

- **Use boundary conditions**
  - $z^* = -1$ when $z = -N$ \ldots \ldots \text{(1)}
  - $z^* = 1$ when $z = -F$ \ldots \ldots \text{(2)}

- **Set up simultaneous equations**

  \[
  -1 = \frac{-aN + b}{N} \implies -N = -aN + b \ldots \ldots \text{(1)}
  \]

  \[
  1 = \frac{-aF + b}{F} \implies F = -aF + b \ldots \ldots \text{(2)}
  \]
Transformation of z: Solve for a and b

\[-N = -aN + b \ldots \ldots (1)\]

\[F = -aF + b \ldots \ldots (2)\]

- Multiply both sides of (1) by -1

\[N = aN - b \ldots \ldots (3)\]

- Add eqns (2) and (3)

\[F + N = aN - aF\]

\[\Rightarrow a = \frac{F + N}{N - F} = \frac{-(F + N)}{F - N} \ldots \ldots (4)\]

- Now put (4) back into (3)
Transformation of z: Solve for a and b

- Put solution for $a$ back into eqn (3)

\[
N = aN - b \quad \text{.......(3)}
\]

\[
\Rightarrow N = \frac{-N(F + N)}{F - N} - b
\]

\[
\Rightarrow b = -N - \frac{-N(F + N)}{F - N}
\]

\[
\Rightarrow b = \frac{-N(F - N) - N(F + N)}{F - N} = \frac{-NF - N^2 - NF + N^2}{F - N} = \frac{-2NF}{F - N}
\]

- So

\[
a = \frac{-(F + N)}{F - N}
\]

\[
b = \frac{-2FN}{F - N}
\]
What does this mean?

- Original point $z$ in original view volume, transformed into $z^*$ in canonical view volume

$$z^* = \frac{az + b}{-z}$$

- where

$$a = \frac{-(F + N)}{F - N}$$

$$b = \frac{-2FN}{F - N}$$
Homogenous Coordinates

- Want to express projection transform as 4x4 matrix
- Previously, homogeneous coordinates of 
  \[ P = (P_x, P_y, P_z) \Rightarrow (P_x, P_y, P_z, 1) \]
- Introduce arbitrary scaling factor, \( w \), so that 
  \[ P = (wP_x, wP_y, wP_z, w) \]  \( \text{(Note: } w \text{ is non-zero)} \)
- For example, the point \( P = (2, 4, 6) \) can be expressed as 
  - \( (2, 4, 6, 1) \)
  - or \( (4, 8, 12, 2) \) where \( w = 2 \)
  - or \( (6, 12, 18, 3) \) where \( w = 3 \), or....
- To convert from homogeneous back to ordinary coordinates, 
  first divide all four terms by \( w \) and discard 4th term
Perspective Projection Matrix

- Recall Perspective Transform

\[(x^*, y^*, z^*) = \left( \frac{N}{x - z}, \frac{N}{y - z}, \frac{az + b}{-z} \right)\]

- We have:

\[x^* = x \frac{N}{-z}, \quad y^* = y \frac{N}{-z}, \quad z^* = \frac{az + b}{-z}\]

- In matrix form:

\[
\begin{pmatrix}
N & 0 & 0 & 0 \\
0 & N & 0 & 0 \\
0 & 0 & a & b \\
0 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
w x \\
w y \\
w z \\
w
\end{pmatrix} =
\begin{pmatrix}
w N x \\
w N y \\
w (az + b) \\
-w z
\end{pmatrix} \Rightarrow
\begin{pmatrix}
x \frac{N}{-z} \\
y \frac{N}{-z} \\
\frac{az + b}{-z} \\
1
\end{pmatrix}
\]

Perspective Transform Matrix | Original vertex | Transformed Vertex | Transformed Vertex after dividing by 4\textsuperscript{th} term
Perspective Projection Matrix

\[
\begin{pmatrix}
N & 0 & 0 & 0 \\
0 & N & 0 & 0 \\
0 & 0 & a & b \\
0 & 0 & -1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
wP_x \\
wP_y \\
wP_z \\
w \\
\end{pmatrix}
= \begin{pmatrix}
wNP_x \\
wNP_y \\
w(aP_z + b) \\
-wP_z \\
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
x \\
y \\
z \\
1 \\
\end{pmatrix}
\]

\[
a = \frac{-(F + N)}{F - N} \quad b = \frac{-2FN}{F - N}
\]

- In perspective transform matrix, already solved for \(a\) and \(b\):
- So, we have transform matrix to transform \(z\) values
Perspective Projection

- Not done yet!! Can now transform z!
- Also need to transform the $x = (\text{left, right})$ and $y = (\text{bottom, top})$ ranges of viewing frustum to [-1, 1]
- Similar to glOrtho, we need to translate and scale previous matrix along x and y to get final projection transform matrix

- We translate by
  - $-(\text{right} + \text{left})/2$ in $x$
  - $-(\text{top} + \text{bottom})/2$ in $y$

- Scale by:
  - $2/(\text{right} - \text{left})$ in $x$
  - $2/(\text{top} - \text{bottom})$ in $y$
Perspective Projection

- Translate along x and y to line up center with origin of CVV
  - \(-(\text{right} + \text{left})/2\) in x
  - \(-(\text{top} + \text{bottom})/2\) in y

- Multiply by translation matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & -(\text{right} + \text{left})/2 \\
0 & 1 & 0 & -(\text{top} + \text{bottom})/2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Line up centers along x and y
Perspective Projection

- To bring view volume size down to size of CVV, scale by:
  - $\frac{2}{\text{right} - \text{left}}$ in $x$
  - $\frac{2}{\text{top} - \text{bottom}}$ in $y$

- Multiply by scale matrix:

$$
\begin{pmatrix}
\frac{2}{\text{right} - \text{left}} & 0 & 0 & 0 \\
0 & \frac{2}{\text{top} - \text{bottom}} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

Scale size down along $x$ and $y$
Perspective Projection Matrix

\[
\begin{pmatrix}
\frac{2}{right - left} & 0 & 0 & 0 \\
0 & \frac{2}{top - bottom} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\times
\begin{pmatrix}
1 & 0 & 0 & -(right + left) / 2 \\
0 & 1 & 0 & -(top + bottom) / 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\times
\begin{pmatrix}
N & 0 & 0 & 0 \\
0 & N & 0 & 0 \\
0 & 0 & a & b \\
0 & 0 & -1 & 0 \\
\end{pmatrix}
\]

glFrustum(left, right, bottom, top, N, F)  \quad N = \text{near plane}, \ F = \text{far plane}
After perspective transformation, viewing frustum volume is transformed into canonical view volume.
Geometric Nature of Perspective Transform

a) Lines through eye map into lines parallel to z axis after transform
b) Lines perpendicular to z axis map to lines perp to z axis after transform
Normalization Transformation

- Original clipping volume
- Original object
- COP
- Z = -x
- Z = x
- Z = -far
- Z = -near
- X = -1
- X = 1
- Z = 1
- Z = -1

distorted object projects correctly

new clipping volume
References