Points, Scalars and Vectors

- Points, vectors defined relative to a coordinate system
- **Point**: Location in coordinate system
- Example: Point (5,4)
- Cannot add or scale points
Vectors

- Magnitude
- Direction
- **NO** position
- Can be added, scaled, rotated
- CG vectors: 2, 3 or 4 dimensions
Vector-Point Relationship

- Subtract 2 points = vector
  \[ \mathbf{v} = Q - P \]
- Point + vector = point
  \[ P + \mathbf{v} = Q \]
Vector Operations

- Define vectors
  \[ \mathbf{a} = (a_1, a_2, a_3) \]
  \[ \mathbf{b} = (b_1, b_2, b_3) \]

Then vector addition:

\[ \mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \]
Vector Operations

- Define scalar, $s$
- Scaling vector by a scalar
  \[ \mathbf{a}s = (a_1s, a_2s, a_3s) \]

Note vector subtraction:

\[ \mathbf{a} - \mathbf{b} = (a_1 + (-b_1), a_2 + (-b_2), a_3 + (-b_3)) \]
Vector Operations: Examples

- **Scaling vector by a scalar**
  \[ \mathbf{as} = (a_1s, a_2s, a_3s) \]

- **Vector addition:**
  \[ \mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \]

- For example, if \( \mathbf{a} = (2,5,6) \) and \( \mathbf{b} = (-2,7,1) \) and \( s = 6 \), then
  \[ \mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) = (0,12,7) \]
  \[ \mathbf{as} = (a_1s, a_2s, a_3s) = (12,30,36) \]
Affine Combination

- Given a vector
  \[ \mathbf{a} = (a_1, a_2, a_3, \ldots, a_n) \]
  \[ a_1 + a_2 + \ldots + a_n = 1 \]

- Affine combination: Sum of all components = 1

- Convex affine = affine + no negative component
  i.e
  \[ a_1, a_2, \ldots, a_n = \text{non-negative} \]
Magnitude of a Vector

- **Magnitude of** \( \mathbf{a} \)

\[
|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 \ldots + a_n^2}
\]

- **Normalizing a vector (unit vector)**

\[
\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\text{vector}}{\text{magnitude}}
\]

- **Note magnitude of normalized vector = 1. i.e**

\[
\sqrt{a_1^2 + a_2^2 \ldots + a_n^2} = 1
\]
Magnitude of a Vector

- Example: if \( \mathbf{a} = (2, 5, 6) \)

- Magnitude of \( \mathbf{a} \)
  \[
  |\mathbf{a}| = \sqrt{2^2 + 5^2 + 6^2} = \sqrt{65}
  \]

- Normalizing \( \mathbf{a} \)
  \[
  \hat{\mathbf{a}} = \left( \frac{2}{\sqrt{65}}, \frac{5}{\sqrt{65}}, \frac{6}{\sqrt{65}} \right)
  \]
Convex Hull

- Smallest convex object containing $P_1, P_2, \ldots, P_n$
- Formed by “shrink wrapping” points
Dot Product (Scalar product)

- Dot product,

\[ d = \mathbf{a} \cdot \mathbf{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + \ldots + a_n \cdot b_n \]

- For example, if \( \mathbf{a} = (2, 3, 1) \) and \( \mathbf{b} = (0, 4, -1) \) then

\[ a \cdot b = (2 \times 0) + (3 \times 4) + (1 \times -1) = 0 + 12 - 1 = 11 \]
Properties of Dot Products

- Symmetry (or commutative):
  \[ a \cdot b = b \cdot a \]

- Linearity:
  \[ (a + c) \cdot b = a \cdot b + c \cdot b \]

- Homogeneity:
  \[ (sa) \cdot b = s(a \cdot b) \]

- And
  \[ |b^2| = b \cdot b \]
Angle Between Two Vectors

\[ \mathbf{b} = (|\mathbf{b}| \cos \phi_b, |\mathbf{b}| \sin \phi_b) \]
\[ \mathbf{c} = (|\mathbf{c}| \cos \phi_c, |\mathbf{c}| \sin \phi_c) \]
\[ \mathbf{b} \cdot \mathbf{c} = |\mathbf{b}| |\mathbf{c}| \cos \theta \]

Sign of \( \mathbf{b} \cdot \mathbf{c} \):

- \( \mathbf{b} \cdot \mathbf{c} > 0 \)
- \( \mathbf{b} \cdot \mathbf{c} < 0 \)
- \( \mathbf{b} \cdot \mathbf{c} = 0 \)
Angle Between Two Vectors

- **Problem**: Find angle b/w vectors $\mathbf{b} = (3,4)$ and $\mathbf{c} = (5,2)$

- **Step 1**: Find magnitudes of vectors $\mathbf{b}$ and $\mathbf{c}$
  
  $|\mathbf{b}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$
  
  $|\mathbf{c}| = \sqrt{5^2 + 2^2} = \sqrt{29}$

- **Step 2**: Normalize vectors $\mathbf{b}$ and $\mathbf{c}$
  
  $\hat{\mathbf{b}} = \left(\frac{3}{5}, \frac{4}{5}\right)$
  $\hat{\mathbf{c}} = \left(\frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right)$
Angle Between Two Vectors

- **Step 3:** Find angle as dot product \( \hat{\mathbf{b}} \cdot \hat{\mathbf{c}} \)

\[
\hat{\mathbf{b}} \cdot \hat{\mathbf{c}} = \left( \frac{3}{5}, \frac{4}{5} \right) \cdot \left( \frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}} \right)
\]

\[
\hat{\mathbf{b}} \cdot \hat{\mathbf{c}} = \frac{15}{5\sqrt{29}} + \frac{8}{5\sqrt{29}} = \frac{23}{5\sqrt{29}} = 0.85422
\]

- **Step 4:** Find angle as inverse cosine

\[
\theta = \cos(0.85422) = 31.326°
\]
Standard Unit Vectors

Define

\[ \mathbf{i} = (1,0,0) \]
\[ \mathbf{j} = (0,1,0) \]
\[ \mathbf{k} = (0,0,1) \]

So that any vector,

\[ \mathbf{v} = (a,b,c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \]
Cross Product (Vector product)

If
\[ \mathbf{a} = (a_x, a_y, a_z) \]
\[ \mathbf{b} = (b_x, b_y, b_z) \]

Then
\[ \mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y) \mathbf{i} - (a_x b_z - a_z b_x) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k} \]

Remember using determinant

Note: \( \mathbf{a} \times \mathbf{b} \) is perpendicular to \( \mathbf{a} \) and \( \mathbf{b} \)
Cross Product

**Note:** $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
Cross Product (Vector product)

Calculate \( \mathbf{a} \times \mathbf{b} \) if \( \mathbf{a} = (3,0,2) \) and \( \mathbf{b} = (4,1,8) \)

\[ \mathbf{a} = (3,0,2) \quad \mathbf{b} = (4,1,8) \]

Using determinant

\[
\begin{vmatrix}
  i & j & k \\
  3 & 0 & 2 \\
  4 & 1 & 8 \\
\end{vmatrix}
\]

Then

\[ \mathbf{a} \times \mathbf{b} = (0 - 2)i - (24 - 8)j + (3 - 0)k 
\]

\[ = -2i - 16j + 3k \]
Normal for Triangle using Cross Product Method

plane \quad \mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0

\mathbf{n} = (\mathbf{p}_2 - \mathbf{p}_0) \times (\mathbf{p}_1 - \mathbf{p}_0)

normalize \quad \mathbf{n} \leftarrow \frac{\mathbf{n}}{|\mathbf{n}|}

Note that right-hand rule determines outward face
Newell Method for Normal Vectors

- Problems with cross product method:
  - Calculation difficult by hand, tedious
  - If 2 vectors almost parallel, cross product is small
  - Numerical inaccuracy may result

- Proposed by Martin Newell at Utah (teapot guy)
  - Uses formulae, suitable for computer
  - Compute during mesh generation
  - Robust!
Newell Method Example

- Example: Find normal of polygon with vertices $P_0 = (6,1,4)$, $P_1=(7,0,9)$ and $P_2 = (1,1,2)$

- Using simple cross product:
  
  $$((7,0,9)-(6,1,4)) \times ((1,1,2)-(6,1,4)) = (2,-23,-5)$$

 ![Diagram of the polygon with vertices $P_0 = (6,1,4)$, $P_1=(7,0,9)$, and $P_2 = (1,1,2)$ and vectors $P_1 - P_0$, $P_2 - P_0$. The cross product is calculated using the vertices.](image)
Newell Method for Normal Vectors

- Formulae: Normal $N = (m_x, m_y, m_z)$

\[
m_x = \sum_{i=0}^{N-1} \left( y_i - y_{\text{next}(i)} \right) \left( z_i + z_{\text{next}(i)} \right)
\]

\[
m_y = \sum_{i=0}^{N-1} \left( z_i - z_{\text{next}(i)} \right) \left( x_i + x_{\text{next}(i)} \right)
\]

\[
m_z = \sum_{i=0}^{N-1} \left( x_i - x_{\text{next}(i)} \right) \left( y_i + y_{\text{next}(i)} \right)
\]
Newell Method for Normal Vectors

- Calculate x component of normal

\[ m_x = \sum_{i=0}^{N-1} (y_i - y_{next(i)})(z_i + z_{next(i)}) \]

\[ m_x = (1)(13) + (-1)(11) + (0)(6) \]
\[ m_x = 13 - 11 + 0 \]
\[ m_x = 2 \]

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
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<td>6</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>P1</td>
<td>7</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>P2</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>P0</td>
<td>6</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>
Newell Method for Normal Vectors

- Calculate y component of normal

\[
m_y = \sum_{i=0}^{N-1} (z_i - z_{next(i)}) (x_i + x_{next(i)})
\]

\[
m_y = (-5)(13) + (7)(8) + (-2)(7)
\]

\[
m_y = -65 + 56 - 14
\]

\[
m_y = -23
\]
Newell Method for Normal Vectors

- Calculate z component of normal

\[ m_z = \sum_{i=0}^{N-1} (x_i - x_{next(i)})(y_i + y_{next(i)}) \]

\[
m_z = (-1)(1) + (6)(1) + (-5)(2)
\]

\[
m_z = -1 + 6 - 10
\]

\[
m_z = -5
\]

Note: Using Newell method yields same result as Cross product method (2, -23, -5)
Finding Vector Reflected From a Surface

- \( a \) = original vector
- \( n \) = normal vector
- \( r \) = reflected vector
- \( m \) = projection of \( a \) along \( n \)
- \( e \) = projection of \( a \) orthogonal to \( n \)

Note: \( \Theta_1 = \Theta_2 \)

\[
e = a - m
\]

\[
r = e - m
\]

\[
\Rightarrow r = a - 2m
\]
Forms of Equation of a Line

- Two-dimensional forms of a line
  - **Explicit:** \( y = mx + h \)
  - **Implicit:** \( ax + by + c = 0 \)
  - **Parametric:**
    \[
    \begin{align*}
    x(\alpha) &= \alpha x_0 + (1-\alpha)x_1 \\
    y(\alpha) &= \alpha y_0 + (1-\alpha)y_1
    \end{align*}
    \]

- Parametric form of line
  - More robust and general than other forms
  - Extends to curves and surfaces
Convexity

- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object.
References