Computer Graphics (CS 543)  
Lecture 3 (part 2): Linear Algebra for Graphics (Points, Scalars, Vectors)  

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Points, Scalars and Vectors

- Points, vectors defined relative to a coordinate system
- Example: Point (5,4)
Vectors

- Magnitude
- Direction
- **NO** position
- Can be added, scaled, rotated
- CG vectors: 2, 3 or 4 dimensions
Points

- Location in coordinate system
- Cannot add or scale
- Subtract 2 points = vector
Vector-Point Relationship

- Diff. b/w 2 points = vector
  \[ \mathbf{v} = Q - P \]

- point + vector = point
  \[ \mathbf{v} + P = Q \]
Vector Operations

- Define vectors
  \[ \mathbf{a} = (a_1, a_2, a_3) \]
  \[ \mathbf{b} = (b_1, b_2, b_3) \]

Then vector addition:

\[ \mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \]
Vector Operations

- Define scalar, $s$
- Scaling vector by a scalar
  \[ \mathbf{a}s = (a_1s, a_2s, a_3s) \]

\[ \mathbf{a} - \mathbf{b} = (a_1 + (-b_1), a_2 + (-b_2), a_3 + (-b_3)) \]

Note vector subtraction:
Vector Operations: Examples

- Scaling vector by a scalar
  \[ \mathbf{as} = (a_1 s, a_2 s, a_3 s) \]

- Vector addition:
  \[ \mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \]

For example, if \( \mathbf{a} = (2, 5, 6) \) and \( \mathbf{b} = (-2, 7, 1) \) and \( s = 6 \), then

\[ \mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) = (0, 12, 7) \]

\[ \mathbf{as} = (a_1 s, a_2 s, a_3 s) = (12, 30, 36) \]
Affine Combination

● Given a vector

\[ \mathbf{a} = (a_1, a_2, a_3, \ldots, a_n) \]

\[ a_1 + a_2 + \ldots + a_n = 1 \]

● Affine combination: Sum of all components = 1

\[ a_1, a_2, \ldots, a_n = non-negative \]

● Convex affine = affine + no negative component
Convex Hull

- Smallest convex object containing $P_1, P_2, \ldots, P_n$
- Formed by “shrink wrapping” points
Magnitude of a Vector

- **Magnitude of \( \mathbf{a} \)**

  \[
  | \mathbf{a} | = \sqrt{a_1^2 + a_2^2 \ldots + a_n^2}
  \]

- **Normalizing a vector (unit vector)**

  \[
  \hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\text{vector}}{\text{magnitude}}
  \]

- **Note magnitude of normalized vector = 1. i.e**

  \[
  \sqrt{a_1^2 + a_2^2 \ldots + a_n^2} = 1
  \]
Dot Product (Scalar product)

- Dot product,
  \[ d = \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \]

- For example, if \( \mathbf{a}=(2,3,1) \) and \( \mathbf{b}=(0,4,-1) \) then
  \[ \mathbf{a} \cdot \mathbf{b} = (2 \times 0) + (3 \times 4) + (1 \times -1) \]
  \[ = 0 + 12 - 1 = 11 \]
Properties of Dot Products

- Symmetry (or commutative):
  \[ a \cdot b = b \cdot a \]

- Linearity:
  \[ (a + c) \cdot b = a \cdot b + c \cdot b \]

- Homogeneity:
  \[ (sa) \cdot b = s(a \cdot b) \]

- And
  \[ |b^2| = b \cdot b \]
Angle Between Two Vectors

\[ \mathbf{b} = (|b| \cos \phi_b, |b| \sin \phi_b) \]

\[ \mathbf{c} = (|c| \cos \phi_c, |c| \sin \phi_c) \]

\[ \mathbf{b} \cdot \mathbf{c} = |\mathbf{b}| |\mathbf{c}| \cos \theta \]

Sign of \( \mathbf{b} \cdot \mathbf{c} \):

- \( \mathbf{b} \cdot \mathbf{c} > 0 \)
- \( \mathbf{b} \cdot \mathbf{c} = 0 \)
- \( \mathbf{b} \cdot \mathbf{c} < 0 \)
Angle Between Two Vectors

- Find the angle b/w the vectors $\mathbf{b} = (3,4)$ and $\mathbf{c} = (5,2)$
Angle Between Two Vectors

- Find the angle b/w the vectors $\mathbf{b} = (3,4)$ and $\mathbf{c} = (5,2)$
  - $|\mathbf{b}| = 5$, $|\mathbf{c}| = 5.385$
  - $\hat{\mathbf{b}} = \left(\frac{3}{5}, \frac{4}{5}\right)$ \hspace{1cm} $\hat{\mathbf{b}} \cdot \hat{\mathbf{c}} = 0.85422 = \cos \theta$

  \[ \theta = 31.326^\circ \]
Standard Unit Vectors

Define

\[ \mathbf{i} = (1, 0, 0) \]
\[ \mathbf{j} = (0, 1, 0) \]
\[ \mathbf{k} = (0, 0, 1) \]

So that any vector,

\[ \mathbf{v} = (a, b, c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \]
Cross Product (Vector product)

If
\[ \mathbf{a} = (a_x, a_y, a_z) \quad \mathbf{b} = (b_x, b_y, b_z) \]

Then
\[ \mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y) \mathbf{i} - (a_x b_z - a_z b_x) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k} \]

Remember using determinant
\[
\begin{vmatrix}
  i & j & k \\
  a_x & a_y & a_z \\
  b_x & b_y & b_z \\
\end{vmatrix}
\]

Note: \( \mathbf{a} \times \mathbf{b} \) is perpendicular to \( \mathbf{a} \) and \( \mathbf{b} \)
**Cross Product**

**Note:** $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
Cross Product

Calculate $\mathbf{a} \times \mathbf{b}$ if $\mathbf{a} = (3,0,2)$ and $\mathbf{b} = (4,1,8)$
Cross Product

Calculate \( \mathbf{a} \times \mathbf{b} \) if \( \mathbf{a} = (3,0,2) \) and \( \mathbf{b} = (4,1,8) \)

\[
\mathbf{a} \times \mathbf{b} = -2\mathbf{i} - 16\mathbf{j} + 3\mathbf{k}
\]
Finding Vector Reflected From a Surface

- \( a \) = original vector
- \( n \) = normal vector
- \( r \) = reflected vector
- \( m \) = projection of \( a \) along \( n \)
- \( e \) = projection of \( a \) orthogonal to \( n \)

Note: \( \theta_1 = \theta_2 \)

\[
\begin{align*}
r &= e - m \\
e &= a - m \\
\Rightarrow r &= a - 2m
\end{align*}
\]
Lines

- Consider all points of the form
  - $P(\alpha) = P_0 + \alpha \mathbf{d}$
  - **Line**: Set of all points that pass through $P_0$ in direction of vector $\mathbf{d}$
Parametric Form

- Two-dimensional forms of a line
  - **Explicit:** \( y = mx + h \)
  - **Implicit:** \( ax + by + c = 0 \)
  - **Parametric:**
    \[
    x(\alpha) = \alpha x_0 + (1-\alpha)x_1 \\
    y(\alpha) = \alpha y_0 + (1-\alpha)y_1
    \]

- Parametric form of line
  - More robust and general than other forms
  - Extends to curves and surfaces
Convexity

- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object.
Curves and Surfaces

- Curves: 1-parameter non-linear functions of the form $P(\alpha)$
- Surfaces are formed from two-parameter functions $P(\alpha, \beta)$
  - Linear functions give planes and polygons
References