Appendix to: Logic and computation in a lambda calculus with intersection and union types (LPAR 2010)

Daniel J. Dougherty
Worcester Polytechnic Institute, USA

Luigi Liquori
INRIA, France

1 The functions $\mathcal{F}$ and $\mathcal{G}$

This is a supplement to the above-named paper [DL10], which defines a λ-calculus *à la* Church corresponding to Curry-style type assignment to an untyped λ-calculus with intersection and union types. Space constraints precluded presenting the complete definition of the functions $\mathcal{F}$ and $\mathcal{G}$ used there: on the next two pages the definitions are given in their entirety.

References

[DL10] Daniel J. Dougherty, Luigi Liquori. Logic and computation in a lambda calculus with intersection and union types. *Proc. 16th International Conference on Logic for Programming Artificial Intelligence and Reasoning (LPAR)* 2010.

$$\mathcal{F}\left(\dfrac{x@\iota{:}\sigma \in \Gamma}{\Gamma \vdash x@\iota : \sigma} \quad (Var) \right) \triangleq \left\{ \begin{array}{l} \dfrac{x{:}\sigma \in B}{B \vdash x : \sigma} \quad (Var) \\ \mathcal{E}(\Gamma) = B \end{array} \right.$$

$$\mathcal{F}\left(\dfrac{}{\Gamma \vdash M@{*} : \omega} \quad (\omega) \right) \triangleq \left\{ \begin{array}{l} \dfrac{}{B \vdash M : \omega} \quad (\omega) \\ \mathcal{E}(\Gamma) = B \end{array} \right.$$

$$\mathcal{F}\left(\dfrac{\mathcal{D}^{\mathsf{t}} : \Gamma, x@\iota{:}\sigma_1 \vdash M@\Delta : \sigma_2}{\Gamma \vdash \lambda x{:}\iota.M@\lambda\iota{:}\sigma_1.\Delta : \sigma_1 \to \sigma_2} \quad (\to I) \right) \triangleq \left\{ \begin{array}{l} \dfrac{\mathcal{F}(\mathcal{D}^{\mathsf{t}}) : B, x{:}\sigma_1 \vdash M' : \sigma_2}{B \vdash \lambda x.M' : \sigma_1 \to \sigma_2} \quad (\to I) \\ \mathcal{E}(\Gamma, x@\iota{:}\sigma_1) = B, x{:}\sigma_1 \ \& \ \mathcal{E}(M@\Delta) = M' \end{array} \right.$$

$$\mathcal{F}\left(\dfrac{\begin{array}{c} \mathcal{D}_1^{\mathsf{t}} : \Gamma \vdash M@\Delta_1 : \sigma_1 \to \sigma_2 \\ \mathcal{D}_2^{\mathsf{t}} : \Gamma \vdash N@\Delta_2 : \sigma_1 \end{array}}{\Gamma \vdash M\,N@\Delta_1\,\Delta_2 : \sigma_2} \quad (\to E) \right) \triangleq \left\{ \begin{array}{l} \dfrac{\begin{array}{c} \mathcal{F}(\mathcal{D}_1^{\mathsf{t}}) : B \vdash M' : \sigma_1 \to \sigma_2 \\ \mathcal{F}(\mathcal{D}_2^{\mathsf{t}}) : B \vdash N' : \sigma_1 \end{array}}{B \vdash M'\,N' : \sigma_2} \quad (\to E) \\ \mathcal{E}(\Gamma) = B \ \& \ \mathcal{E}(M@\Delta_1) = M' \ \& \ \mathcal{E}(N@\Delta_2) = N' \end{array} \right.$$

$$\mathcal{F}\left(\dfrac{\begin{array}{c} \mathcal{D}_1^{\mathsf{t}} : \Gamma \vdash M@\Delta_1 : \sigma_1 \\ \mathcal{D}_2^{\mathsf{t}} : \Gamma \vdash M@\Delta_2 : \sigma_2 \end{array}}{\Gamma \vdash M@\langle \Delta_1\ ,\ \Delta_2 \rangle : \sigma_1 \wedge \sigma_2} \quad (\wedge I) \right) \triangleq \left\{ \begin{array}{l} \dfrac{\begin{array}{c} \mathcal{F}(\mathcal{D}_1^{\mathsf{t}}) : B \vdash M' : \sigma_1 \\ \mathcal{F}(\mathcal{D}_2^{\mathsf{t}}) : B \vdash M' : \sigma_2 \end{array}}{B \vdash M' : \sigma_1 \wedge \sigma_2} \quad (\wedge I) \\ \mathcal{E}(\Gamma) = B \ \& \ \mathcal{E}(M@\langle \Delta_1\ ,\ \Delta_2 \rangle) = M' \end{array} \right.$$

$$\mathcal{F}\left(\dfrac{\mathcal{D}^{\mathsf{t}} : \Gamma \vdash M@\Delta : \sigma_1 \wedge \sigma_2}{\Gamma \vdash M@\mathsf{pr_i}\Delta : \sigma_i} \quad \begin{array}{c} i = 1,2 \\ (\wedge E_i) \end{array} \right) \triangleq \left\{ \begin{array}{l} \dfrac{\mathcal{F}(\mathcal{D}^{\mathsf{t}}) : B \vdash M' : \sigma_1 \wedge \sigma_2}{B \vdash M' : \sigma_i} \quad \begin{array}{c} i = 1,2 \\ (\wedge E_i) \end{array} \\ \mathcal{E}(\Gamma) = B \ \& \ \mathcal{E}(M@\Delta) = M' \end{array} \right.$$

$$\mathcal{F}\left(\dfrac{\mathcal{D}_1^{\mathsf{t}} : \Gamma \vdash M@\Delta : \sigma_i}{\Gamma \vdash M@\mathsf{in_i}\Delta : \sigma_1 \vee \sigma_2} \quad \begin{array}{c} i = 1,2 \\ (\vee I_i) \end{array} \right) \triangleq \left\{ \begin{array}{l} \dfrac{\mathcal{F}(\mathcal{D}^{\mathsf{t}}) : B \vdash M' : \sigma_i}{B \vdash M' : \sigma_1 \vee \sigma_2} \quad \begin{array}{c} i = 1,2 \\ (\vee I_i) \end{array} \\ \mathcal{E}(\Gamma) = B \ \& \ \mathcal{E}(M@\mathsf{in_i}\Delta) = M' \end{array} \right.$$

$$\mathcal{F}\left(\dfrac{\begin{array}{c} \mathcal{D}_1^{\mathsf{t}} : \Gamma, x@\iota_1{:}\sigma_1 \vdash M@\Delta_1 : \sigma_3 \\ \mathcal{D}_2^{\mathsf{t}} : \Gamma, x@\iota_2{:}\sigma_2 \vdash M@\Delta_2 : \sigma_3 \\ \mathcal{D}_3^{\mathsf{t}} : \Gamma \vdash N@\Delta_3 : \sigma_1 \vee \sigma_2 \end{array}}{\Gamma \vdash M[N/x]@\left[\begin{array}{c} \lambda\iota_1{:}\sigma_1.\Delta_1, \\ \lambda\iota_2{:}\sigma_2.\Delta_2 \end{array} \right] \Delta_3 : \sigma_3} \quad (\vee E) \right) \triangleq \left\{ \begin{array}{l} \dfrac{\begin{array}{c} \mathcal{F}(\mathcal{D}_1^{\mathsf{t}}) : B, x{:}\sigma_1 \vdash M'' : \sigma_3 \\ \mathcal{F}(\mathcal{D}_2^{\mathsf{t}}) : B, x{:}\sigma_2 \vdash M'' : \sigma_3 \\ \mathcal{F}(\mathcal{D}_3^{\mathsf{t}}) : B \vdash N' : \sigma_1 \vee \sigma_2 \end{array}}{B \vdash M' : \sigma_3} \quad (\vee E) \\ \mathcal{E}(\Gamma) = B \ \& \ \mathcal{E}(M[N/x]@\left[\begin{array}{c} \lambda\iota_1{:}\sigma_1.\Delta_1, \\ \lambda\iota_2{:}\sigma_2.\Delta_2 \end{array} \right] \Delta_3) = M' \\ \mathcal{E}(M@\Delta_{1,2}) = M'' \ \& \ \mathcal{E}(N@\Delta_3) = N' \end{array} \right.$$

Figure 1: The Function $\mathcal{F}$.

$$\mathcal{G}\left(\dfrac{x{:}\sigma\in B}{B\vdash x:\sigma}\quad(\mathit{Var})\right) \triangleq \begin{cases} \dfrac{x@\iota{:}\sigma\in\Gamma}{\Gamma\vdash x@\iota:\sigma}\quad(\mathit{Var}) \\ \mathcal{E}(\Gamma)=B \quad \iota \text{ is fresh}\end{cases}$$

$$\mathcal{G}\left(\dfrac{}{B\vdash M':\omega}\quad(\omega)\right) \triangleq \begin{cases} \dfrac{}{\Gamma\vdash M@{*}:\omega}\quad(\omega) \\ \mathcal{E}(\Gamma)=B\ \&\ \mathcal{E}(M)=M'\end{cases}$$

$$\mathcal{G}\left(\dfrac{\mathcal{D}^{\mathsf{u}}: B,x{:}\sigma_1\vdash M':\sigma_2}{B\vdash \lambda x.M':\sigma_1\to\sigma_2}\quad(\to I)\right) \triangleq \begin{cases} \dfrac{\mathcal{G}(\mathcal{D}^{\mathsf{u}}):\Gamma,x@\iota{:}\sigma_1\vdash M@\Delta:\sigma_2}{\Gamma\vdash(\lambda x{:}\iota.M)@(\lambda\iota{:}\sigma_1.\Delta):\sigma_1\to\sigma_2}\quad(\to I) \\ \mathcal{E}(\Gamma,x@\iota{:}\sigma_1)=B,x{:}\sigma_1\ \&\ \mathcal{E}(M@\Delta)=M'\end{cases}$$

$$\mathcal{G}\left(\dfrac{\begin{array}{c}\mathcal{D}^{\mathsf{u}}_1: B\vdash M':\sigma_1\to\sigma_2\\ \mathcal{D}^{\mathsf{u}}_2: B\vdash N':\sigma_1\end{array}}{B\vdash M'\,N':\sigma_2}\quad(\to E)\right) \triangleq \begin{cases} \dfrac{\begin{array}{c}\mathcal{G}(\mathcal{D}^{\mathsf{u}}_1):\Gamma\vdash M@\Delta_1:\sigma_1\to\sigma_2\\ \mathcal{G}(\mathcal{D}^{\mathsf{u}}_2):\Gamma\vdash N@\Delta_2:\sigma_1\end{array}}{\Gamma\vdash M\,N@\Delta_1\,\Delta_2:\sigma_2}\quad(\to E) \\ \mathcal{E}(\Gamma)=B\ \&\ \mathcal{E}(M@\Delta_1)=M'\ \&\ \mathcal{E}(N@\Delta_2)=N'\end{cases}$$

$$\mathcal{G}\left(\dfrac{\begin{array}{c}\mathcal{D}^{\mathsf{u}}_1: B\vdash M':\sigma_1\\ \mathcal{D}^{\mathsf{u}}_2: B\vdash M':\sigma_2\end{array}}{B\vdash M':\sigma_1\wedge\sigma_2}\quad(\wedge I)\right) \triangleq \begin{cases} \dfrac{\begin{array}{c}\mathcal{G}(\mathcal{D}^{\mathsf{u}}_1):\Gamma\vdash M@\Delta_1:\sigma_1\\ \mathcal{G}(\mathcal{D}^{\mathsf{u}}_2):\Gamma\vdash M@\Delta_2:\sigma_2\end{array}}{\Gamma\vdash M@\langle\Delta_1\,,\ \Delta_2\rangle:\sigma_1\wedge\sigma_2}\quad(\wedge I) \\ \mathcal{E}(\Gamma)=B\ \&\ \mathcal{E}(M@\langle\Delta_1\,,\ \Delta_2\rangle)=M'\end{cases}$$

$$\mathcal{G}\left(\dfrac{\mathcal{D}^{\mathsf{u}}: B\vdash M':\sigma_1\wedge\sigma_2}{B\vdash M':\sigma_i}\quad\begin{array}{c}i=1,2\\(\wedge E_i)\end{array}\right) \triangleq \begin{cases} \dfrac{\mathcal{G}(\mathcal{D}^{\mathsf{u}}):\Gamma\vdash M@\Delta:\sigma_1\wedge\sigma_2}{\Gamma\vdash M@\mathsf{pr_i}\Delta:\sigma_i}\quad\begin{array}{c}i=1,2\\(\wedge E_i)\end{array} \\ \mathcal{E}(\Gamma)=B\ \&\ \mathcal{E}(M@\Delta)=M'\end{cases}$$

$$\mathcal{G}\left(\dfrac{\mathcal{D}^{\mathsf{u}}: B\vdash M':\sigma_i}{B\vdash M':\sigma_1\vee\sigma_2}\quad\begin{array}{c}i=1,2\\(\vee I_i)\end{array}\right) \triangleq \begin{cases} \dfrac{\mathcal{G}(\mathcal{D}^{\mathsf{u}}):\Gamma\vdash M@\Delta:\sigma_i}{\Gamma\vdash M@\mathsf{in_i}\Delta:\sigma_1\vee\sigma_2}\quad\begin{array}{c}i=1,2\\(\vee I_i)\end{array} \\ \mathcal{E}(\Gamma)=B\ \&\ \mathcal{E}(M@\mathsf{in_i}\Delta)=M'\end{cases}$$

$$\mathcal{G}\left(\dfrac{\begin{array}{c}\mathcal{D}^{\mathsf{u}}_1: B,x{:}\sigma_1\vdash M':\sigma_3\\ \mathcal{D}^{\mathsf{u}}_2: B,x{:}\sigma_2\vdash M':\sigma_3\\ \mathcal{D}^{\mathsf{u}}_3: B\vdash N':\sigma_1\vee\sigma_2\end{array}}{B\vdash M'[N'/x]:\sigma_3}\quad(\vee E)\right) \triangleq \begin{cases} \dfrac{\begin{array}{c}\mathcal{G}(\mathcal{D}^{\mathsf{u}}_1):\Gamma,x@\iota_1{:}\sigma_1\vdash M@\Delta_1:\sigma_3\\ \mathcal{G}(\mathcal{D}^{\mathsf{u}}_2):\Gamma,x@\iota_2{:}\sigma_2\vdash M@\Delta_2:\sigma_3\\ \mathcal{G}(\mathcal{D}^{\mathsf{u}}_3):\Gamma\vdash N@\Delta_3:\sigma_1\vee\sigma_2\end{array}}{\Gamma\vdash M[N/x]@\left[\begin{array}{c}\lambda\iota_1{:}\sigma_1.\Delta_1,\\ \lambda\iota_2{:}\sigma_2.\Delta_2\end{array}\right]\Delta_3:\sigma_3}\quad(\vee E) \\ \mathcal{E}(\Gamma)=B\ \&\ \mathcal{E}(M[N/x]@\left[\begin{array}{c}\lambda\iota_1{:}\sigma_1.\Delta_1,\\ \lambda\iota_2{:}\sigma_2.\Delta_2\end{array}\right]\Delta_3)=M'[N'/x] \\ \mathcal{E}(M@\Delta_{1,2})=M'\ \&\ \mathcal{E}(N@\Delta_3)=N'\end{cases}$$

Figure 2: The Function $\mathcal{G}$.