This may be as good a place as any to repeat that when you submit a home assignment or an exam **you should show all work.** Otherwise, when a result seems to appear “out of blue sky” without showing the work that led to it, this is a prescription for trouble. It is a bad idea even when the result is correct, but when it is not—and we all do mistakes—I do not know if it was only a trivial calculation error in which case you lose few if any points. Seeing no work, I shall assume the procedure you used to get the result was wrong. This can make a serious difference in the final grade for the problem. See my solution for part 3(b) below. You may think this is extreme, but it is worth your time—and saves mine.

**One more recurrence**

1. Solve the second-order linear recurrence

   \[ x_n - \frac{1}{3}x_{n-1} - \frac{2}{3}x_{n-2} = n, \quad n \geq 2, \quad x_0 = a, \quad x_1 = b. \]

**Solution:** By now this should be—and usually was—quite routine. Most followed the route of ‘solve the homogeneous equation–guess a particular solution (which had to be nudged to a second-degree polynomial in \(n\))–use initial values to determine coefficients from the homogeneous solution.’ In that case you get the homogeneous equation \(3z^2 - 2 = 0\), which has the roots \(-\frac{2}{3}\) and 1. The particular solution comes in at \(\frac{3}{10}n^2 + \frac{27}{50}n\), hence the general solution is \(A + B(-2/3)^n + \frac{3}{10}n^2 + \frac{27}{50}n\). Plugging in the initial values results in \(A = \frac{3}{5}a + \frac{3}{5}b - \frac{126}{250}\) and \(B = \frac{3}{5}a - \frac{1}{3}b + \frac{126}{250}\).

An alternative method proceeds through two steps of homogenizing the original equation, so a fourth-order homogeneous recurrence needs to be handled. The resultant characteristic equation is a quartic: \(3z^4 - 7z^3 + 3z^2 + 3z - 2 = 0\). Quartic equations are rarely delightful, but this one makes a good try, in having the roots \(-\frac{2}{5}\) and 1, the latter three times. The general solution is therefore of the form \(\alpha(-\frac{2}{5})^n + A + Bn + Cn^2\). We need four initial values – but only two are given. The way out of this difficulty is to iterate the recurrence twice, to calculate two more initial values \(x_2\) and \(x_3\), which are given by \(x_2 = \frac{b}{3} + \frac{2a}{3} + 2\), \(x_3 = \frac{7b}{3} + \frac{2a}{3} + \frac{11}{3}\). Solving the equations for \(\alpha\), \(A\), \(B\) and \(C\) is not much more pleasant than usual – and the results coincide, naturally.

**Generating Functions**

2. Here is a different way to derive formulas for the sums of the powers of the natural numbers, i.e., formulas for \(\sum_{k=1}^{n} k^a\), for \(a=1,2,3\), etc.

   The idea is to use the correspondence between a sequence obtained from \(\{a_i\}\) by multiplying each term by the same polynomial, say \(P(i)\), and the OGF of this new sequence, when expressed in
terms of the OGF of the original sequence \( a(z) \). According to what we show in class—and it can be read from the text (p. 85 table)—the OGF of \( \{P(i) a_i\} \) is \( P(zD)a(z) \), where \( D \) is the differentiation operator.

Derive then these formulas, \( \sum_{k=1}^{n} k^a \), for \( a=1,2,3, \) etc. The more the better.

**Guidance:** Use for \( a_i \) the all-ones sequence. To compute \( \sum_{n} k^a \), evaluate \( zD\frac{1}{1-z} \), and at the end divide once more by \( 1 - z \), to obtain the partial sums. Finally, extract the coefficient of \( z^n \) from the GF you obtained.

**Solution:** The idea was to give you an exercise in manipulating generating functions and extraction of coefficients, and in particular, some opportunity to use the correspondence between operations on sequences and on their GFs that was shown in class, but not much was done to illustrate it. You really only needed to follow the above guidance—which happens to be complete—to get it right, but it is, admittedly, quite a bit of a drag once you go beyond the first couple of steps. The only case where people made innocent mistakes is when misinterpreting the operator \( zD^2 \) or any higher power: it means \( zDzD \), and not any other of the possible 6 permutations (yes, six: 4 elements can be permuted 24 ways, but since we have two identical pairs we need to divide twice by 2!...).

Let us see: when \( a = 1 \) we need to compute \( zD(1-z)^{-1} \) and get \( z(1-z)^{-2} \). Then tack on another \( (1-z)^{-1} \) for the partial sum operator and compute:

\[
[z^a]z(1-z)^{-3} = [z^{a-1}](1-z)^{-3} = \binom{-3}{n-1}(-1)^{n-1} = \binom{n+1}{n-1} = \frac{n+1}{n-1} = \frac{n(n+1)}{2}.
\]

The third transition from the right used the “negation of the upper parameter” formula for binomial coefficients which is always permitted: \( \binom{-1}{s} = \binom{-(s-1)}{s} \).

For \( a = 2 \) we begin with \( zD[z(1-z)^{-2}] = z(1+z)(1-z)^{-3} \), and then need to extract

\[
[z^n]z(1+z)(1-z)^{-4} = \left[ z^n \right] \frac{z^2 + z^3}{(1-z)^4} = [z^{n-1}](1-z)^{-4} + [z^{n-2}](1-z)^{-4} = \binom{-4}{n-1}(-1)^{n-1} + \binom{-4}{n-2}(-1)^{n-2} = \binom{n+1}{n-1} + \binom{n+1}{n-2} = \frac{n(n+1)}{6} \binom{n+2}{n}.
\]

One can go one or two steps beyond that working by hand, but I protect my sanity by using a symbolic calculation system. Let us see what Maple produces (I added some interpolations, mainly the expressions \( \sum_{k=1}^{n} k^a \)).

We begin by defining our building block:

\[
> f0 := z \rightarrow 1/(1-z) ;
\]

\[
f0 := z \rightarrow \frac{1}{1-z}
\]
The following is a peculiarity of Maple, due to the way it represents internally expressions and functions.

```maple
f1 := unapply(z*a, z);
```

`genfunc` is a package of procedures that handle formal power series. In particular, `rgf_expand` extracts general coefficients from rational GFs. This is admittedly very limited, but happens to be all we need here. To my knowledge, there are no general-purpose procedure that does this over arbitrary functions. (Though the various Taylor series functions will be happy to produce any reasonable prefix of the sequence of coefficients. Moreover, this is an efficient procedure.)

```maple
with(genfunc):

factor(normal(rgf_expand(f1(z)*f0(z), z, n)));
```

```
\sum_{k=1}^{n} k = \frac{1}{2} n(n+1)
```

I was surprised to see most of you (all?) were not familiar with this result...

```maple
a := normal(diff(f3(z), z));
a := -\frac{1+z}{(-1+z)^3}
```
> f4 := unapply(z*a,z);
> f4 := z \rightarrow \frac{z(1+11z+11z^2+z^3)}{(-1+z)^5}

> factor(normal(rgf_expand(f4(z)*f0(z),z,n)));
\[ \sum_{k=1}^{n} k^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1) \]

And one more:
> a := normal(diff(f4(z),z));
> a := 1 + 26z + 26z^3 + z^4 + 66z^2 \\
> f5 := unapply(z*a,z);
> f5 := z \rightarrow z(1+26z+26z^3+z^4+66z^2) \\
> factor(normal(rgf_expand(f5(z)*f0(z),z,n)));
\[ \sum_{k=1}^{n} k^5 = \frac{1}{12} n^2(2n^2+2n-1)(n+1)^2 \]

3. Let \( S \) be the operator “prefix summation” on a sequence. This means, that given an arbitrary sequence \( \{ a_i \}_{i \geq 0} \), the operator creates the sequence \( \{ S a_i \}_{i \geq 0} \), that has the terms \( S a_i \equiv \sum_{j=0}^{i} a_j \).

We define now the sequence \( S^0 a_i, S a_i, S^2 a_i, \ldots \), using the recursive definition \( S^n a_i \equiv S[S^{n-1}a_i] \).

The first term in the sequence refers to the null operation: \( S^0 a_i = a_i \).

(a) Let \( a(z) \) be the OGF of \( \{ a_i \}_{i \geq 0} \). Find \( A_n(z) \), the OGF of \( \{ S^n a_i \}_{i \geq 0} \).

**Solution:** Here you needed to do very little beyond commenting that since the “prefix summation” simply performs the partial sum operation, and we saw that that operation corresponds to multiplying the OGF by \( \frac{1}{1-z} \), then

\[ A_n(z) \equiv \sum_{i \geq 0} (S^n a_i) z^i = \frac{a(z)}{(1-z)^n}. \]

**Comment:** Several of you were misled by the notation \( S^n a_i \), thinking it means \( S^n(a_i) \). In fact, if parentheses were to be used there at all, it would be written as \( (S^n) a_i \), to signify that this is the \( i \)th term in a sequence called \( S^n a \). The operator \( S \) operates on the entire sequence, not on individual elements.

Show that

\[ S^n a_i = \sum_{j=0}^{i} \binom{n+j-1}{j} a_{i-j}. \]
(b) Using straightforward mathematical induction (over the recursive definition of $S^n$).

**Solution:** The natural basis is to pick $n = 0$, and this holds:

$$S^0 a_i = \sum_{j=0}^{i} \binom{j-1}{j} a_{i-j} = a_i,$$

since the binomial coefficient $\binom{j-1}{j}$ vanishes for all values of $j$ except $j = 0$, when it is 1.

**Note:** Several of you decided that a good basis is at $n = 2$. I do not know why. However, if the requirement is to prove the property holds for all $n \geq 0$, you still need to prove for $n = 0, 1$, explicitly, so where is the gain?

We shall assume it holds up to a certain $n$, and set to prove the claim for the next term in the sequence. There are several ways to go about it, and the following one is what seemed to come naturally as well (and I did not try to look for a shorter route, which I assume exists):

$$S^{n+1} a_i \equiv \sum_{k=0}^{i} S^n a_k \quad \text{definition of the S operator}$$

$$= \sum_{k=0}^{i} \sum_{j=0}^{k} \binom{n+j-1}{j} a_{k-j} \quad \text{Substituting from the induction hypothesis}$$

$$= \sum_{j=0}^{i} \binom{n+j-1}{j} \sum_{k=j}^{i} a_{k-j} \quad \text{Change order of summation, isolate the } a_{k-j}$$

$$= \sum_{j=0}^{i} \binom{n+j-1}{j} \sum_{k=0}^{i-j} a_k \quad \text{Shift the index of summation } k$$

$$= \sum_{k=0}^{i} a_k \sum_{j=0}^{i-k} \binom{n+j-1}{j} \quad \text{Change the order of summation again}$$

$$= \sum_{k=0}^{i} a_k \binom{n+i-k}{i-k} \quad \text{Use the summation formula } \sum_{t=0}^{v} \binom{u+t}{u} = \binom{u+v+1}{u}$$

$$= \sum_{j=0}^{i} \binom{n+j}{j} a_{i-j} \quad \text{Change index of summation } k = i - j.$$  

And the above is in the desired form. Hence the claim is proved. I now see I did it once (p. 26 in my Analysis book) somewhat differently, but doing essentially the same operations.

(c) By using the OGF $A_n(z)$ you computed in part (a), and extracting coefficients.

**Solution:** This one is simpler, nearly immediate. Since $A_n(z) = \frac{a(z)}{(1-z)^n}$, we treat the RHS as a product of the functions $a(z)$ and $(1-z)^{-n}$ and use the convolution operation:

$$S^n a_i = [z^i] (1-z)^{-n} a(z) = \sum_{j=0}^{i} [z^j] (1-z)^{-n} \times [z^{i-j}] a(z) = \sum_{j=0}^{i} \binom{-n}{j} (-1)^j a_{i-j}$$

and negating the upper index in the binomial coefficient completes the calculation.