Recurr ences

This homework is still all about recurrences exclusively.

1. In class I displayed how certain nonhomogeneous equations can be “homogenized” at relatively low cost. It turns out that for the linear equations with constant coefficients, this can be done exactly for those non-homogeneous parts which we have seen before: either a pure exponential (e.g. $e^n$) or a polynomial in $n$ or a mixture—additive or multiplicative—of both.

   Let me recapitulate with the simplest case, which is a paradigm for them all.

   Suppose we have the equation $x_{n+1} - tx_n = c$, for known constants $c$ and $t$. We rewrite the equation with a shift of the index, and subtract:
   
   $x_{n+1} - tx_n = c$
   
   $-[x_n - tx_{n-1} = c]$

   to get the homogeneous second order equation $x_{n+1} - (t+1)x_n + tx_{n-1} = 0$. Note it still has constant coefficients

   (a) Now consider the following recurrence, given in problem 2 of Homework 4:
   
   $x_n = x_{n-1} + ax_{n-2} + bn, \quad n \geq 2,$

   and homogenize it as well. For the special values $a = 6$ and $b = 5$, solve separately the two equations, and show the solutions coincide.

   (b) For extra bonus, Prove the following assertion: if the RHS of a linear recurrence with constant coefficients is a polynomial in $n$ of degree $r$, homogenization of the equation leads to an increase of its order by $r+1$.

   Solution: (a) For some reason I neglected to suggest specific initial values here, in addition to specifying the $a$ and $b$ values. This would have simplified showing the actual identity of the solutions...

   As it is, being left with $x_0$ and $x_1$, we shall recruit computers for help.

   The homogenization needs here two steps, where in the first we get rid of the linear (in $n$) term:
   
   $x_n = x_{n-1} + ax_{n-2} + bn$
   
   $-[x_{n+1} = x_n + ax_{n-1} + b(n + 1)]$

   and we find the still-nonhomogeneous equation of third order:

   $2x_n - x_{n+1} = (1 - a)x_{n-1} + ax_{n-2} - b,$

   and we create the same scheme again

   $2x_n - x_{n+1} = (1 - a)x_{n-1} + ax_{n-2} - b$
   
   $-[2x_{n+1} - x_{n+2} = (1 - a)x_n + ax_{n-1} - b],$
We ended with a fourth-order recurrence, which is valid for all $n \geq 2$, but needs four initial values. The way to get them—assuming two were given—is to iterate the original recurrence twice and generate the two additional values. This fourth order recurrence gives rise to a quartic characteristic equation, which we only present for the value of $a$ given, 6: $z^4 - 3z^3 - 3z^2 + 11z - 6 = 0$. By inspection this has 1 as a root, and dividing by $z - 1$ we find the cubic equation $z^3 - 2z^2 - 5z + 6 = 0$, which also has $z = 1$ for a root, and a further division produces the quadratic $z^2 - z - 6 = 0$ that has the roots 3 and -2.

Hence the general solution here is $x_n = A(-2)^n + B\cdot 3^n + Cn + D$. (Remember the rule that when $\beta$ is a root of the characteristic equation of order $r$, the coefficient of $\beta^n$ in the general solution is a polynomial in $n$ of degree $r - 1$, Here $r = 2$ for $\beta = 1$, and naturally we omit $1^n = 1$).

In Homework 4 we found the solution to the nonhomeogenous equation given by $x_n = Au^n + Bv^n - \frac{u}{n} - \frac{b+2a}{a}$. For the given $a$ we find $u = -2$, $v = 3$

To go beyond this we called on Maple, and asked it to do the calculations (my input is in tty font, and Maple’s responses are typeset as displayed equations):

We first solve for $A$ and $B$ of the solution of the inhomogeneous equation, in terms of the initial values:

```plaintext
> e1 := {x0 = A+B-65/36, x1 = -2*A+3*B - 5/6 - 65/36 }:
> solve(e1, {A, B});
```

```plaintext
{A = 3/5 \times 0 - 1/5 \times 1 + 5/9, B = 1/5 \times 1 + 2/5 \times 0 + 5/4}
```

Next we need to define $x_2$ and $x_3$ in terms of the “real” initial values. First define the recurrence as a function:

```plaintext
> x := n -> x(n-1)+6\times x(n-2)+5\times n;
```

and provide it with initial values (Maple needs to be told such things explicitly):

```plaintext
> x(0) := x0: x(1) := x1:
```

Now we take the general solution of the homogeneous equation, and define it as a function,

```plaintext
> y := n -> S*(-2)^n + T*3^n +C*n+D;
```
which we apply for the four initial index values:

> e2 := map(y, [0, 1, 2, 3]);

\[ e2 := [S + T + D, -2S + 3T + C + D, 4S + 9T + 2C + D, -8S + 27T + 3C + D] \]

Then we set up the four equations for these values:


\[ e3 := \{S + T + D = x0, -2S + 3T + C + D = x1, 4S + 9T + 2C + D = 6x0 + x1 + 10, \]
\[ -8S + 27T + 3C + D = 6x0 + 7x1 + 25\} \]

(note that the definition of the recurrence as a function produces here all the value in terms of the initial values \(x_0\) and \(x_1\)). Finally we solve for all the unknown coefficients — obtaining the expected result:

> solve(e3, \{S, T, C, D\});

\[ \{S = \frac{3}{5}x0 - \frac{1}{5}x1 + \frac{5}{9}, T = \frac{1}{5}x1 + \frac{2}{5}x0 + \frac{5}{4}, C = -\frac{5}{6}, D = -\frac{65}{36}\}. \]

(In truth I edited here the Maple output: it returns the solution as a set of values which would usually be permuted, with respect to the order I specified — \(S, T, C, D\) — so I rearranged them).

2. Exercise 2.27 of the text (S&F). For completeness, I bring it here. Notice that there are here two questions:

Give initial conditions \(a_0, a_1\) for which the solution of

\[ a_n = 5a_{n-1} - 6a_{n-2}, \quad n \geq 2, \]

is \(a_n = 2^n\). Are there initial conditions for which the solution is \(a_n = 2^n - 1\)?

**Solution:** Most of you got it right with apparent ease. The equation is homogeneous, and gives rise to the characteristic equation \(z^2 - 5z + 6 = 0\), which factors to \((z - 2)(z - 3) = 0\). The roots are 2 and 3, and the general solution is \(a_n = A2^n + B3^n\). The \(A\) and \(B\) are to be solved via the equations

\[ a_0 = A + B; \quad a_1 = 2A + 3B. \]
Here we reverse the situation: we ask for $a_0$ and $a_1$ for which the solution is $A = 1$, $B = 0$, which by substitution produces $a_0 = 1$, $a_1 = 2$.

**Note:** Naturally this is what you get if you evaluate $a_n = 2^n$ with $n = 0, 1$. Some of you did just this – without solving the equation – but in such a complicated way, usually via higher order elements of $\{a_n\}$, that I tend to believe they did not even notice that this is what they do.

The answer to the second question is a denial: we have the general solution, and no, it does not look like the desired function $a_n = 2^n - 1$, regardless of the initial conditions.

3. Consider the recurrence given in Exercise 2.27 (should have written 2.31) of the text (S&F):

$$a_n = a_{n-1} - a_{n-2}, \quad n \geq 2, \quad a_0 = 0, \quad a_1 = 1.$$  

This is a second order linear recurrence with constant coefficients, so we know how to solve it. It is even easier to iterate it a few times and understand the solution. Show the two “methods” agree.

**Solution:** This one presented some difficulties. Indeed it takes some searching (I surely needed some) to find a way to show that the intuitive method – iterate the recurrence until you realize its nature:

$$\{a_j\} = \{0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1 \cdots\}$$  

(1)

gives the same result as the usual formal method of going through the solution of the characteristic equation.

The latter is here $z^2 - z + 1 = 0$, with roots $(1 \pm \sqrt{3})/2$, and solving for the indicated two coefficients provides the complete solution

$$a_n = \frac{1}{i\sqrt{3}} \left( \left( \frac{1+i\sqrt{3}}{2} \right)^n - \left( \frac{1-i\sqrt{3}}{2} \right)^n \right).$$  

(2)

Are these solution identical? Yes, they are: to show this you 

(a) substitute $n = (0, 1, 2, 3, 4, 5)$ in the general solution (2), and obtain the first six entries in (1) [you need of course only the last four of these].

(b) You show that the function given in (2) satisfies $a_n = a_{n+6}$. In fact, the two somewhat stronger claims are rather simple to show:

$$\left( \frac{1+i\sqrt{3}}{2} \right)^n = \left( \frac{1+i\sqrt{3}}{2} \right)^{n+6}, \quad \left( \frac{1-i\sqrt{3}}{2} \right)^n = \left( \frac{1-i\sqrt{3}}{2} \right)^{n+6}.$$  

For example:

$$\left( 1 + i\sqrt{3} \right)^6 = 1 + 6i\sqrt{3} + 15(i\sqrt{3})^2 + 20(i\sqrt{3})^3 + 15(i\sqrt{3})^4 + 6(i\sqrt{3})^5 + (i\sqrt{3})^6$$
\[
1 + i(6 - 20 \cdot 3 + 6 \cdot 9)\sqrt{3} - 15 \cdot 3 + 15 \cdot 9 - 27 = 64.
\]

Note: The above was written before I plowed into your rendition of the homework. There I found a very elegant solution of the unpleasant part of this problem. One of you made the observation that since \(\cos(\pi/3) = 1/2\), and \(\sin(\pi/3) = \sqrt{3}/2\), the following identities hold (using a formula for the trigonometric function in terms of the exponential you saw in Homework 2):

\[
\frac{1 + i\sqrt{3}}{2} = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = e^{i\pi/3}, \quad \frac{1 - i\sqrt{3}}{2} = \cos \frac{\pi}{3} - i \sin \frac{\pi}{3} = e^{-i\pi/3},
\]

which leads to

\[
a_n = \frac{1}{i\sqrt{3}} \left( \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)^n - \left( \cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right)^n \right) = \frac{1}{i\sqrt{3}} \left( e^{in\pi/3} - e^{-in\pi/3} \right) = \frac{2i}{i\sqrt{3}} \frac{\sin \frac{n\pi}{3}}{\sin \frac{\pi}{3}}.
\]

The periodicity is obvious here, since the sine function cycles every \(2\pi\), and the \(a_n\) values are very easy to compute.